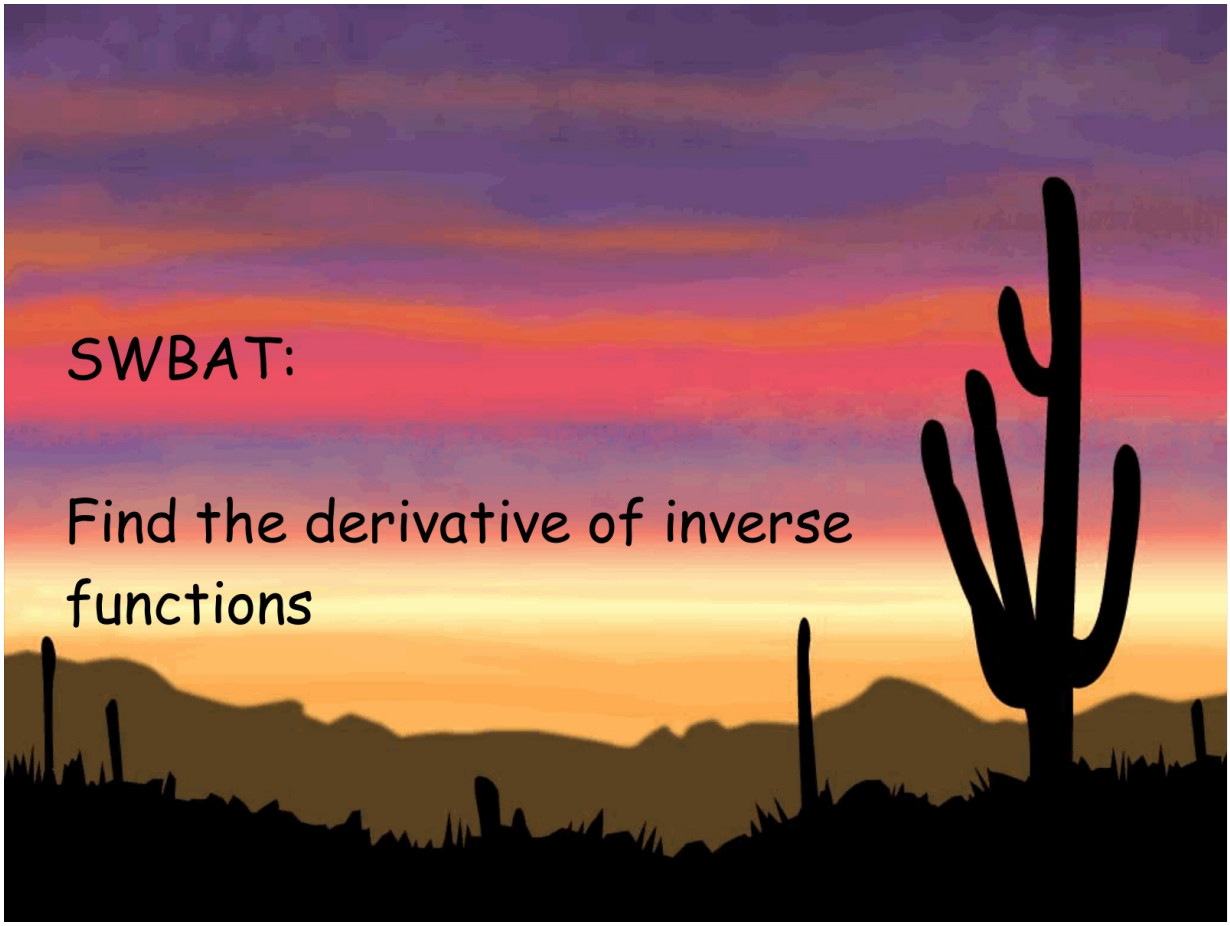

Inverse functions

If f(x) and g(x) are inverse functions:

$f(g(x)) = x$

domain of f = range of g
range of f = domain of g
$f(a) = b \Rightarrow g(b) = a$

Find g'(x) - the derivative of $f^{-1}(x)$

$$\frac{d}{dx}\left(f(g(x)) = x\right)$$

$$f'(g(x))\,g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))}$$

U. Derivative of an Inverse of a Function

What you are finding: There is probably no topic that confuses students (and teachers) more than inverses. The inverse of a function f is another function f^{-1} that "undoes" what f does. So $f^{-1}(f(x)) = x$. For instance, the inverse of adding 5 is subtracting 5. Start with any number x, add 5, then subtract 5, and you are back to x. Do not confuse the inverse f^{-1} with the reciprocal. $x^{-1} = \frac{1}{x}$ but $(f(x))^{-1} \neq \frac{1}{f(x)}$.

To find the inverse of a function, you replace x with y and y with x. The inverse to the function $y = 4x - 1$ is $x = 4y - 1$ or $y = \frac{x+1}{4}$.

In this section, you are concerned with finding the derivative of the inverse to a function: $\frac{d}{dx}[f(x)]^{-1}$

How to find it: The formula used is: $\frac{dy}{dx} = \frac{1}{f'(y)}$. But what I suggest, rather than memorizing this formula, is to switch x and y to find the inverse, and then take the derivative, using implicit differentiation:

$x = f(y) \Rightarrow 1 = f'(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)}$.

Example 72: Find the derivative of the inverse to $y = x^3$ at $x = 1$.

Example 73: Find the derivative of the inverse to $f(x) = 3x + \sin \pi x$ at $x = 6$.

$g(x) = f^{-1}(x)$ $\quad g(6) = ?$ $\quad g(6) = 2$

$g'(x) = \frac{1}{f'(g(x))}$ $\quad \rightarrow f(?) = 6$

$3x + \sin(\pi x) = 6$

$x = 2$

$g'(6) = \frac{1}{f'(g(6))} \rightarrow g'(6) = \frac{1}{f'(2)} = \frac{1}{3+\pi}$

$f'(x) = 3 + \cos(\pi x)(\pi)$

$f'(2) = 3 + \cos(2\pi)(\pi) = 3 + \pi$

Example 74: (Calc) Find the derivative of the inverse to $f(x) = x + 3\sin x$ at $x = 6$.

$x + 3\sin x = 6$

Example 75: If $f(x) = x^2$ (1^{st} quadrant), write the equation of the tangent line to $f^{-1}(x)$ at $x = 16$.

Example 76: If $f(x) = x^3 + x^2 + x + 1$, write the equation of the tangent line to $f^{-1}(x)$ at $x = 4$.

$g(x) = f^{-1}(x)$

$g'(x) = \frac{1}{f'(g(x))}$

$g'(4) = \frac{1}{f'(g(4))}$

$g(4) = ? \quad \rightarrow g(4) = 1$

$\rightarrow f(?) = 4$

$x^3 + x^2 + x + 1 = 4$

$x = 1$

$g'(4) = \frac{1}{f'(1)}$

$f'(x) = 3x^2 + 2x + 1$

$f'(1) = 3(1)^2 + 2(1) + 1 = 6$

$g'(4) = \frac{1}{6}$ (slope)

$(4, 1)$ (point)

$y - 1 = \frac{1}{6}(x - 4)$