CHAPTER

12 Probability

American artist Carmen Lomas Garza (b 1948) created this color etching, *Lotería–Primera Tabla* (1972), which shows one game card in the traditional Mexican game *lotería*. As in bingo, a caller randomly selects one image that may appear on the game cards and players try to cover an entire row, column, diagonal, or all four corners of their game cards. The chance of winning many games, including *lotería*, can be calculated using probability.

OBJECTIVES

In this chapter you will

- learn about randomness and the definition of probability
- count numbers of possibilities to determine probabilities
- determine expected values of random variables
- discover how numbers of combinations relate to binomial probabilities

LESSON

12.1

Randomness and Probability

It's choice—not chance—that determines your destiny.

JEAN NIDETCH

"It isn't fair," complains Noah. "My car insurance rates are much higher than yours." Rita replies, "Well, Noah, that's because insurance companies know the chances are good that it will cost them less to insure me."

How much do you think Demolition Derby drivers pay for auto insurance?

Insurance companies can't know for sure what kind of driving record you will have. So they use the driving records for people of your age group, gender, and prior driving experience to determine your chances of an accident and, therefore, your insurance rates. This is just one example of how probability theory and the concept of randomness affect your life.

Career CONNECTION

An actuary uses mathematics to solve financial problems. Actuaries often work for insurance, consulting, and investment companies. They use probability, statistics, and risk theory to decide the cost of a company's employee benefit plan, the cost of a welfare plan, and how much funding an insurance company will need to pay for expected claims.

Probability theory was developed in the 17th century as a means of determining the fairness of games, and it is still used to make sure that casino customers lose more money than they win. Probability is also important in the study of sociological and natural phenomena.

At the heart of probability theory is randomness. Rolling a die, flipping a coin, drawing a card, and spinning a game-board spinner are examples of **random processes.** In a random process, no individual outcome is predictable, even though the long-range pattern of many individual outcomes often is predictable.

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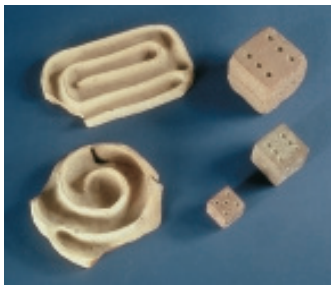
History
CONNECTION

Many games are based on random outcomes, or chance. Paintings and excavated material from Egyptian tombs show that games using *astragali* were established by the time of the First Dynasty, around 3500 B.C.E. An *astragalus* is a small bone in the foot and was used in games resembling modern dice games. In the Ptolemaic dynasty (323–30 B.C.E.) games with 6-sided dice became common in Egypt. People in ancient Greece made icosahedral (20-sided) and other polyhedral dice. The Romans were such enthusiastic dice players that laws were passed forbidding gambling except in certain seasons.



Playing dice, or *tesserae,* was a popular game during the Roman Empire (ca. 1st century B.C.E.–5th century C.E.). The mosaic on the left depicts three men playing *tesserae.* The photo on the right shows ancient Gallo-Roman counters and dice, dating from the second half of the 1st century B.C.E.

Investigation
Flip a Coin

You will need

- a coin

In this investigation you will explore the predictability of random outcomes. You will use a familiar random process, the flip of a coin.

Step 1 Imagine you are flipping a **fair** coin, one that is equally likely to land heads or tails. Without flipping a coin, record a random arrangement of ten H's and T's, as though you were flipping a coin ten times. Label this Sequence A.

Step 2 Now flip a coin ten times and record the results on a second line. Label this Sequence B.

Step 3 How is Sequence A different from the result of your coin flips? Make at least two observations.

Step 4 Find the longest string of consecutive H's or T's in Sequence A. Do the same for Sequence B. Then find the second-longest string. Record these lengths for each person in the class as tally marks in a table.

Longest string	Sequence A	Sequence B	2nd-longest string	Sequence A	Sequence B
1			1		
2			2		
3			3		
4			4		
5 or more			5 or more		

Step 5 Count the number of H's in each set. Record the results of the entire class in a table.

Step 6 If you were asked to write a new random sequence of H's and T's, how would it be different from what you recorded in Sequence A?

Number of H's	Sequence A	Sequence B
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

You often use a random process to generate **random numbers.** Over the long run, each number is equally likely to occur, and there is no pattern in any sequence of numbers generated. [▶ See **Calculator Note 1L** to learn how to generate random numbers. ◀]

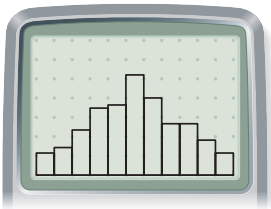
EXAMPLE A

Use a calculator's random-number generator to find the probability of rolling a sum of 6 with a pair of dice.

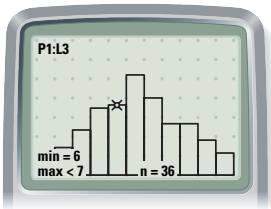
▶ Solution

As you study this solution, follow along on your own calculator. Your results will be slightly different. [▶ To learn how to simulate rolling two dice, see **Calculator Note 12A.** ◀]

To find the probability of the event "the sum is 6," also written as P(sum is 6), simulate a large number of rolls of a pair of dice. First, create a list of 300 random integers from 1 to 6 to simulate 300 tosses of the first die. Store it in list L_1. Store a second list of 300 outcomes in list L_2. Add the two lists to get a list of 300 random sums of two dice. Store it in list L_3.



[2, 13, 1, 0, 70, 10]



[2, 13, 1, 0, 70, 10]

Create a histogram of the 300 entries in the sum list. The calculator screens above show the number of each of the sums from 2 to 12. (Your lists and histogram will show different entries.) Tracing shows that the bin height of the "6" bin is 36. So, out of 300 simulated rolls, $P(\text{sum is } 6) = \frac{36}{300} = .12$.

Repeating the entire process five times gives slightly different results: $\frac{249}{1800} \approx .138$.

Rather than actually rolling a pair of dice 1800 times in Example A, you performed a **simulation,** representing the random process electronically. You can use dice, coins, spinners or electronic random-number generators to simulate trials and explore the probabilities of different **outcomes,** or results.

An **event** is a set of desired outcomes. You might recall that the probability of an event, such as "the sum of two dice is 6," must be a number between 0 and 1. The probability of an event that is certain to happen is 1. The probability of an impossible event is 0. In Example A, you found that P(sum is 6) is approximately .14, or 14%.

Probabilities that are based on trials and observations like this are called **experimental probabilities.** A pattern often does not become clear until you observe a large number of trials. Find your own results for 300 or 1800 simulations of a sum of two dice. How do they compare with the outcomes in Example A?

Sometimes you can determine the **theoretical probability** of an event, without conducting an experiment. To find a theoretical probability, you count the number of ways a desired event can happen and compare this number to the total number of equally likely possible outcomes. Outcomes that are "equally likely" have the same chance of occurring. For example, you are equally likely to flip a head or a tail with a fair coin.

Experimental Probability

If $P(E)$ represents the probability of an event, then

$$P(E) = \frac{\textit{number of occurrences of an event}}{\textit{total number of trials}}$$

Theoretical Probability

If $P(E)$ represents the probability of an event, then

$$P(E) = \frac{\textit{number of different ways the event can occur}}{\textit{total number of equally likely outcomes possible}}$$

How can you calculate the theoretical probability of rolling a sum of 6 with two dice? You might at first think that there are 11 possible sums of the two dice (from 2 through 12), so $P(\text{sum is } 6) = \frac{1}{11}$. But the 11 sums are not equally likely. Imagine that one die is green and the other is white, for example. Then you can get a sum of 5 in four ways:

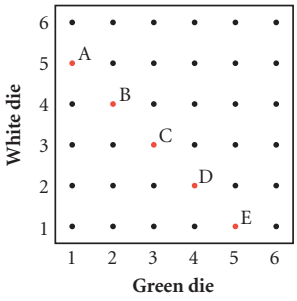
Green	White
1	4
2	3
3	2
4	1

But you get a sum of 12 only if you roll a 6 on both dice. Therefore, a sum of 5 is more likely than a sum of 12.

So, what equally likely outcomes can you use in this situation to find the theoretical probability?

EXAMPLE B Find the theoretical probability of rolling a sum of 6 with a pair of dice.

▶ Solution The possible equally likely outcomes, or sums, when you roll two dice are represented by the 36 grid points in this diagram. The point in the upper-left corner represents a roll of 1 on the first die and 6 on the second die, for a total of 7.



The five possible outcomes with a sum of 6 are labeled A–E in the diagram. Point D, for example, represents an outcome of 4 on the green die and 2 on the white die. What outcome does point A represent?

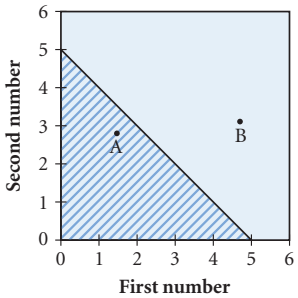
The theoretical probability is the number of ways the event can occur, divided by the number of equally likely events possible. So, $P(\text{sum is } 6) = \frac{5}{36} \approx .1389$, or 13.89%.

Before moving on, compare the experimental and theoretical results of this event. Do you think the experimental probability of an event can vary? How about its theoretical probability?

When you roll dice, the outcomes are whole numbers. What if outcomes could be other kinds of numbers? In those cases, you can often use an area model to find probabilities.

EXAMPLE C What is the probability that any two numbers you select at random between 0 and 6 have a sum that is less than or equal to 5?

▶ Solution Because the two values are no longer limited to integers, counting would be impossible. The possible outcomes are represented by all points within a 6-by-6 square.



In the diagram, point A represents the outcome $1.47 + 2.8 = 4.27$, and point B represents $4.7 + 3.11 = 7.81$. The points in the triangular shaded region are all those with a sum less than or equal to 5. They satisfy the inequality $n_1 + n_2 \leq 5$, where n_1 is the first number and n_2 is the second number. The area of this triangle is $(0.5)(5)(5) = 12.5$. The area of all possible outcomes is $(6)(6) = 36$. The probability is therefore $\frac{12.5}{36} \approx .347$, or 34.7%.

A probability that is found by calculating a ratio of lengths or areas is called a **geometric probability**.

Experimental probabilities can help you estimate a trend if you have enough cases. But obtaining enough data to observe what happens in the long run is not always feasible. Calculating theoretical probabilities can help you predict these trends. In the rest of this chapter, you'll explore different ways to calculate numbers of outcomes in order to find theoretical probabilities.

In skee ball the probability of getting different point values is based on the geometric area of the regions and their distance from the player.

EXERCISES

Practice Your Skills

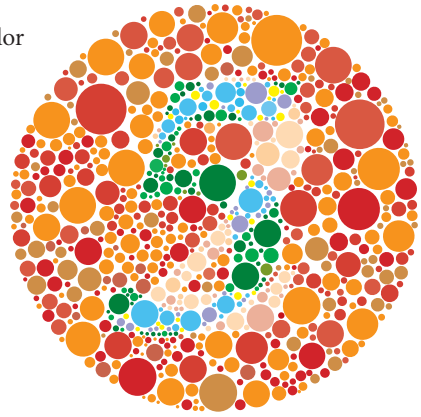
1. Nina has observed that her coach does not coordinate the color of his socks to anything else that he wears. Guessing that the color is a random selection, she records these data during three weeks of observation:

black, white, black, white, black, white, black, red, white, red, white, white, white, black, black

a. What is the probability that he will wear black socks the next day?

b. What is the probability that he will wear white socks the next day?

c. What is the probability that he will wear red socks the next day?



This is a visual test for color blindness, a condition that affects about 8% of men and 0.5% of women. A person with red-green color blindness (the most common type) will see the number 2 more clearly than the number 5 in this image. Dr. Shinobu Ishihara developed this test in 1917.

2. This table shows numbers of students in several categories at Ridgeway High. Find the probabilities described below. Express each answer to the nearest 0.001.

	Male	Female	Total
10th grade	263	249	512
11th grade	235	242	477
12th grade	228	207	435
Total	726	698	1424

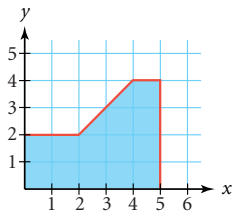
a. What is the probability that a randomly chosen student is female?

b. What is the probability that a randomly chosen student is an 11th grader?

c. What is the probability that a randomly chosen 12th grader is male?

d. What is the probability that a randomly chosen male is a 10th grader?

3. The graph of the shaded area at right shows all possible combinations of two numbers, x and y. Use the graph and basic area formulas to answer each question. Express each answer to the nearest 0.001.



a. What is the probability that x is between 0 and 2?

b. What is the probability that y is between 0 and 2?

c. What is the probability that x is greater than 3?

d. What is the probability that y is greater than 3?

e. What is the probability that $x + y$ is less than 2?

4. Find each probability.

a. Each day, your teacher randomly calls on 5 students in your class of 30. What is the probability you will be called on today?

b. If 2.5% of the items produced by a particular machine are defective, then what is the probability that a randomly selected item will *not* be defective?

c. What is the probability that the sum of two tossed dice will not be 6?

Keymath.com More Practice Your Skills

Reason and Apply

5. To prepare necklace-making kits, three camp counselors pull beads out of a box, one at a time. They discuss the probability that the next bead pulled out of the box will be red. Describe each probability as theoretical or experimental.

a. Claire said that $P(\text{red}) = \frac{1}{2}$, because 15 of the last 30 beads she pulled out were red.

b. Sydney said that $P(\text{red}) = \frac{1}{2}$, because the box label says that 1000 of the 2000 beads are red.

c. Mavis says $P(\text{red}) = \frac{1}{3}$, because 200 of the 600 beads the three of them have pulled out so far have been red.

6. Suppose you are playing a board game for which you need to roll a 6 on a die before you can start playing.

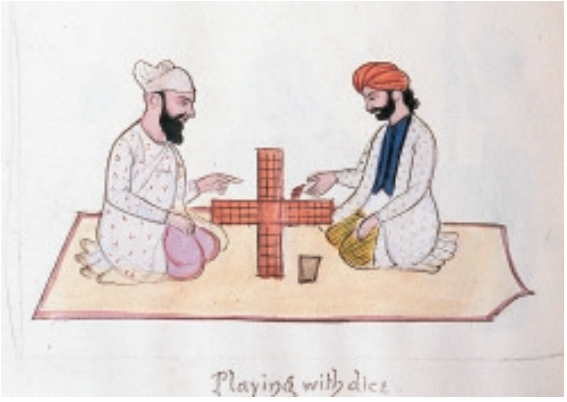
a. Predict the average number of rolls a player should expect to wait before starting to play.

b. Describe a simulation, using random numbers, that you could use to model this problem.

c. Do the simulation ten times and record the number of rolls you need to start playing in each game. (For example, the sequence of rolls 4, 3, 3, 1, 6 means you start playing on the fifth roll.)

d. Find the average number of rolls needed to start during these ten games.

e. Combine your results from 6d with those of three classmates, and approximate the average number of rolls a player should expect to wait.



This watercolor (ca. 1890) by an unknown Indian artist shows men playing dice.

7. Rank i–iii according to the method that will best produce a random integer from 0 to 9. Support your reasoning with complete statements.

i. the number of heads when you drop nine pennies

ii. the length, to the nearest inch, of a standard 9 in. pencil belonging to the next person you meet who has a pencil

iii. the last digit of the page number closest to you after you open a book to a random page and spin it

8. Simulate rolling a fair die 100 times with your calculator's random-number generator. Display the results in a histogram to see the number of 1's, 2's, 3's, and so on. [▶ To learn how to display random numbers in a histogram, see **Calculator Note 12A.** ◀] Do the simulation 12 times.

a. Make a table storing the results of each simulation. Calculate the experimental probability of rolling a 3 after each 100 rolls.

Simulation number	1's	2's	3's	4's	5's	6's	Ratio of 3's	Cumulative ratio of 3's
1								$\frac{?}{100} = \frac{?}{__}$
2								$\frac{?}{200} = \frac{?}{__}$
3								$\frac{?}{300} = \frac{?}{__}$

b. What do you think the long-run experimental probability will be?

c. Make a graph of the cumulative ratio of 3's versus the number of tosses. Plot the points (cumulative number of tosses, cumulative ratio of 3's). Then plot three more points as you extend the domain of the graph to 2400, 3600, and 4800 trials by adding the data from three classmates. Would it make any difference if you were considering 5's instead of 3's? Explain.

d. What is $P(3)$ for this experiment?

e. What do you think the theoretical probability, $P(3)$, should be? Explain.

9. Consider rolling a green die and a white die. The roll (1, 5) is different from (5, 1).

a. How many different outcomes are possible for this two-die experiment?

b. How many different outcomes are possible in which there is a 4 on the green die? Draw a diagram to show the location of these points. What is the probability of this event?

c. How many different outcomes are possible in which there is a 2 or a 3 on the white die? What is the probability of this event?

d. How many different outcomes are possible in which there is an even number on the green die and a 2 on the white die? What is the probability of this event?

10. Find the number of equally likely outcomes of each event described for a two-die roll. Then write the probability of each event.

a. The dice sum to 9.

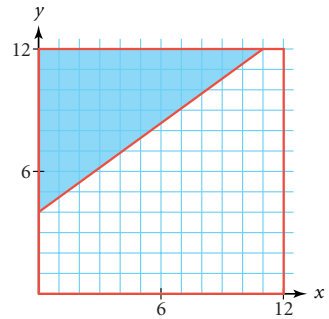
b. The dice sum to 6.

c. The dice have a difference of 1.

d. The sum of the dice is 6, and their difference is 2.

e. The sum of the dice is at most 5.

11. Consider the diagram at right.



a. What is the total area of the square?

b. What is the area of the shaded region?

c. Suppose the horizontal and vertical coordinates are randomly chosen numbers between 0 and 12, inclusive. Over the long run, what ratio of these points will be in the shaded area?

d. What is the probability that any randomly chosen point within the square will be in the shaded area?

e. What is the probability that the randomly chosen point will *not* land in the shaded area?

f. What is the probability that any point randomly selected within the square will land on a specific point? On a specific line?

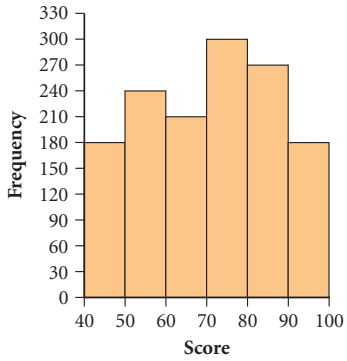
12. Suppose x and y are both randomly chosen numbers between 0 and 8. (The numbers are not necessarily integers.)

a. Write a symbolic statement describing the event that the sum of the two numbers is at most 6.

b. Draw a two-dimensional picture of all possible outcomes, and shade the region described in 12a.

c. Determine the probability of the event described in 12a.

13. Use the histogram at right for 13a–d.



a. Approximate the frequency of scores between 80 and 90.

b. Approximate the sum of all the frequencies.

c. Find P(a score between 80 and 90).

d. Find P(a score that is not between 80 and 90).

14. A 6 in. cube painted on the outside is cut into 27 smaller congruent cubes. Find the probability that one of the smaller cubes, picked at random, will have the specified number of painted faces.

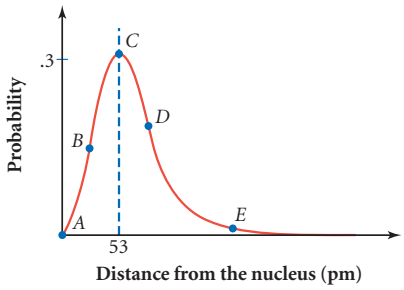
a. exactly one

b. exactly two

c. exactly three

d. no painted face

15. **APPLICATION** Where, in atoms, do electrons reside? The graph at right shows the probability of the electron of a hydrogen atom being at various distances from the nucleus at any given moment. The distances are measured in picometers (pm). A picometer is 1×10^{-12} m. Use the graph to answer these questions.



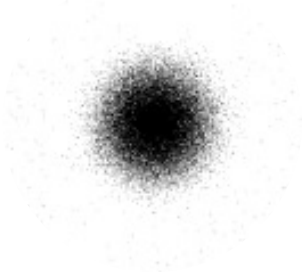
a. At which distance from the nucleus (points A–E) is the probability of finding the electron greatest?

b. At what distance is there zero probability of finding the electron?

c. As the distance from the nucleus increases, describe what happens to the probability of finding an electron.

Science CONNECTION

Electron position is important to scientists because it lays the foundation for understanding chemical changes at the atomic level and determining how likely it is that particular chemical changes will take place. In the early part of the 20th century, scientists thought that electrons orbit the nucleus of an atom much the way the planets orbit the Sun. Danish physicist Niels Bohr (1885–1962) theorized that the electron of a hydrogen atom always orbits at 53 pm from the nucleus. In the mid-1920s, Austrian physicist Erwin Schrödinger (1887–1961) first proposed using a probability model to describe the electron's position. Instead of a predefined orbit, Schrödinger's work led to the electron cloud model.



This electron cloud model shows possible locations of an electron in a hydrogen molecule. The density of points in a particular region indicates the probability that an electron will be located in that area.

Review

16. Expand $(x - y)^4$.

17. Write $\log a - \log b + 2\log c$ as a single logarithmic expression.

18. Solve $\log 2 + \log x = 4$.

19. Consider this system of inequalities:

$$\begin{cases} 3x + y \leq 15 \\ x + 6y \geq -12 \\ -5x + 4y \leq 26 \end{cases}$$

a. Graph the triangle defined by this system.

b. Give the coordinates of the vertices of the triangle in 19a.

c. Find the area of the triangle in 19a.

20. Describe the locus of points equidistant from the line $y = 6$ and the point $(3, 0)$. Then write the polynomial equation in general form.

21. Consider these two sets of data.

i. $\{5, 23, 36, 48, 63\}$ **ii.** $\{112, 115, 118, 119, 121\}$

a. Which set would you expect to have the larger standard deviation? Explain your reasoning.

b. Calculate the mean and the standard deviation of each set.

c. Predict how the mean and the standard deviation of each set will be affected if you multiply every data value by 10. Then do calculations to verify your answer. How do these measures compare to those you found in 21b?

d. Predict how the mean and the standard deviation of each set will be affected if you add 10 to every data value. Then do calculations to verify your answer. How do these measures compare to those you found in 21b?

EXPLORATION

Geometric Probability

French naturalist Georges Louis Leclerc, Comte de Buffon (1707–1788), posed one of the first geometric probability problems: If a coin is tossed randomly onto a floor of congruent tiles, what is the probability that it will land entirely within a tile, not touching any edges? The answer depends on the size of the coin and the size and shape of the tiles. You'll explore this problem through experimentation, then analyze your results.

Activity

The Coin Toss Problem

You will need

- a millimeter ruler
- a penny, a nickel, a dime, and a quarter
- grid paper in different sizes: 20 mm, 30 mm, 40 mm, and 40 mm with 5 mm borders

Work with your group to investigate Buffon's coin toss problem using different-size grids to simulate different-size tiles. For the first three grids, you can assume the borders around each tile have no thickness. Divide up the experimentation among your group members so that you can collect the data more quickly. Record your results in a table like the one below.

	Coin diameter (mm)	20 mm grid	30 mm grid	40 mm grid	40 mm grid with 5 mm borders
Penny					
Nickel					
Dime					
Quarter					

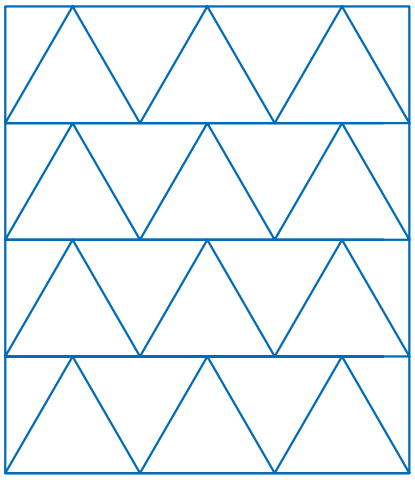
Step 1 Measure the diameter of your coin, and record your answer in millimeters in your table. Toss your coin 100 times, and count the number of times the coin lands entirely within a square, not touching any lines. Record your number of successes, and the experimental probability of success. Collect data from your group members to complete the table.

Step 2 Where must the center of a coin fall in order to have a successful outcome? What is the area of this region for each combination of coin and grid paper? What is the area of each square? How do you account for the borders in the fourth grid paper?

Step 3 Use your answers from Step 2 to calculate the theoretical probability of success for each coin and grid-paper combination. How do these theoretical probabilities compare to your experimental probabilities? If they are significantly different, explain why.

Questions

1. Design a grid, different from any you used in the activity, that has a probability of success of .01, for a coin of your choice.
2. Determine a formula that will calculate the theoretical probability of success, given a coin with diameter d and a grid of squares with side length a and line thickness t.
3. What would be the theoretical probability of success if you tossed a coin with diameter 10 mm onto an infinite grid paper tiled with equilateral triangles with side length 40 mm, arranged as shown below?



IMPROVING YOUR REASONING SKILLS

Beating the Odds

Tracy and Trish have two boxes. One box contains 50 blue marbles; the other box contains 50 red marbles. Tracy will blindfold Trish and place the two boxes in front of her. Trish will pick one marble from one of the boxes. If Trish picks a red marble, she wins. If she picks a blue marble, Tracy wins. Before being blindfolded, Trish requests that she be allowed to distribute the marbles between the boxes in any way she likes. Tracy thinks about the request and says, "Sure, as long as all one hundred marbles are there, what difference could it make?" How should Trish distribute the marbles to have the greatest chance of winning? What would be the probability that Trish will win?

LESSON 12.2

Counting Outcomes and Tree Diagrams

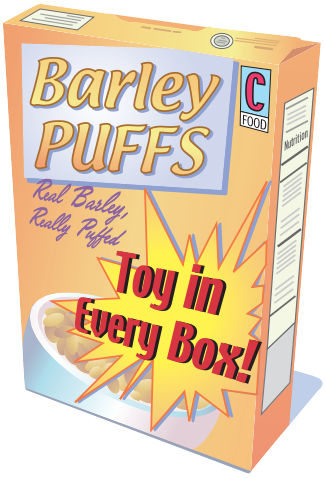
In Lesson 12.1, you determined some theoretical probabilities by finding the ratio of the number of desired outcomes to the number of possible equally likely outcomes. In some cases it can be difficult to count the number of possible or desired outcomes. You can make this easier by organizing information and counting outcomes using a **tree diagram**.

EXAMPLE A

A national advertisement says that every puffed-barley cereal box contains a toy and that the toys are distributed equally. Talya wants to collect a complete set of the different toys from cereal boxes.

a. If there are two different toys, what is the probability that she will find both of them in her first two boxes?

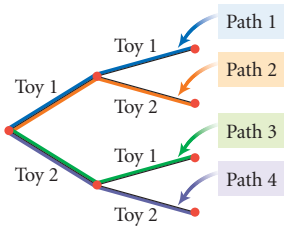
b. If there are three different toys, what is the probability that she will have them all after buying her first three boxes?



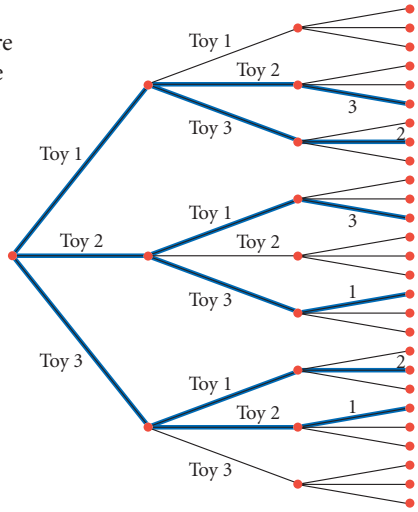
▶ Solution

Draw tree diagrams to organize the possible outcomes.

a. In this tree diagram, the first branching represents the possibilities for the first box and the second branching represents the possibilities for the second box. Thus, the four paths from left to right represent all possibilities for two boxes and two toys. Path 2 and Path 3 contain both toys. If the advertisement is accurate about equal distribution of toys, then the paths are equally likely. So, the probability of getting both toys is $\frac{2}{4}$, or .5.



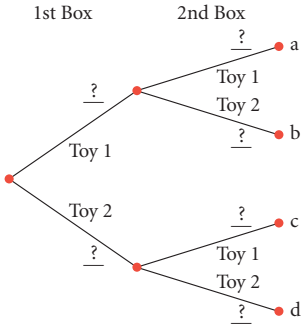
b. This tree diagram shows all the toy possibilities for three boxes. There are 27 possible paths. You can determine this quickly by counting the number of branches on the far right. Six of the 27 paths contain all three toys, as shown. Because the paths are equally likely, the probability of having all three toys is $\frac{6}{27}$, or approximately .222.



Tree diagrams can represent clearly the total number of different outcomes and can help you to identify those paths representing the desired outcomes. Each single branch of the tree represents a **simple event.** A path, or sequence of simple events, is a **compound event.** Tree diagrams can be helpful for organizing complicated situations.

Investigation
The Multiplication Rule

Step 1 On your paper, redraw the tree diagram for Example A, part a. This time, write the probability of each simple event on each branch. Then find the probability of each path.



P(Toy 1 followed by Toy 1) = ?

P(Toy 1 followed by Toy 2) = ?

P(Toy 2 followed by Toy 1) = ?

P(Toy 2 followed by Toy 2) = ?

Step 2 Redraw the tree diagram for Example A, part b. Indicate the probability of each simple event, and also write the probability of each path. What is the sum of the probabilities of all possible paths? What is the sum of the probabilities of the highlighted paths?

Step 3 Suppose the national advertisement in Example A listed four different toys distributed equally in a huge supply of boxes. Draw only as much of a tree diagram as you need to in order to answer these questions:

a. What would be P(Toy 2) in Talya's first box? Talya's second box? Third box? P(any particular toy in any particular box)?

b. In these situations, does the toy she finds in one box influence the probability of there being a particular toy in the next box?

c. One outcome that includes all four toys is Toy 3, followed by Toy 2, followed by Toy 4, followed by Toy 1. What is the probability of this outcome? How many different equally likely outcomes are there?

Step 4 Write a statement explaining how to use the probabilities of a path's branches to find the probability of the path.

Step 5 What is P(obtaining the complete set in the first four boxes)?

In some cases, such as that for four toys and four boxes, a tree diagram with equally likely branches is a lot to draw. In some cases, as you'll see in Example B, a tree with branches of different probabilities may be practical.

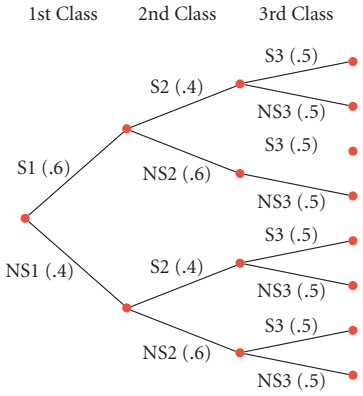
EXAMPLE B

Mr. Roark teaches three classes. Each class has 20 students. His first class has 12 sophomores, his second class has 8 sophomores, and his third class has 10 sophomores. If he randomly chooses one student from each class to participate in a competition, what is the probability that he will select three sophomores?

▶ Solution

You could consider drawing a tree with 20 branches representing the students in the first class. This would split into 20 branches for the second class, and each of these paths would split into 20 branches for the third class. This would be a tree with 8000 paths!

Instead, you can draw two branches for each stage of the selection process. One branch represents a choice of a sophomore (S) and one represents a choice of a nonsophomore (NS). This tree shows all eight possible outcomes. However, the outcomes are not equally likely. For the first class, the probability of choosing a sophomore is $\frac{12}{20} = .6$, and the probability of choosing a nonsophomore is $1 - .6 = .4$. Calculate the probabilities for the second and third classes, and represent them on a tree diagram, as shown.



The uppermost path represents a sophomore being chosen from each class. In the investigation you learned to find the probability of a path by multiplying the probabilities of the branches. So, the probability of choosing three sophomores is (.6)(.4)(.5), or .12.

In Example B, the probability of choosing a sophomore in the second class is the same, regardless of whether a sophomore was chosen in the first class. These events are called **independent.** Events are independent when the occurrence of one has no influence on the occurrence of the other.

Probability of a Path (The Multiplication Rule for Independent Events)

If n_1, n_2, n_3, and so, on represent events along a path, then the probability that this sequence of events will occur can be found by multiplying the probabilities of the events.

$$P(n_1 \text{ and } n_2 \text{ and } n_3 \text{ and } \ldots) = P(n_1) \cdot P(n_2) \cdot P(n_3) \cdot \cdots$$

Are there events that aren't independent? If so, how do you calculate their theoretical probabilities? Mr. Roark's situation provides another example.

EXAMPLE C

Consider the situation from Example B again.

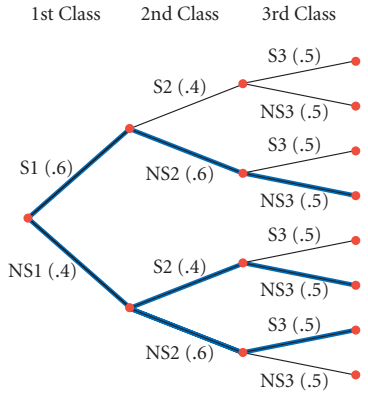
a. What is the probability that Mr. Roark will choose only one sophomore to participate in the competition?

b. Suppose you are a sophomore in Mr. Roark's second class, and the competition rules say that only one sophomore can be on the three-person team. What is the probability that you will be selected?

▶ Solution

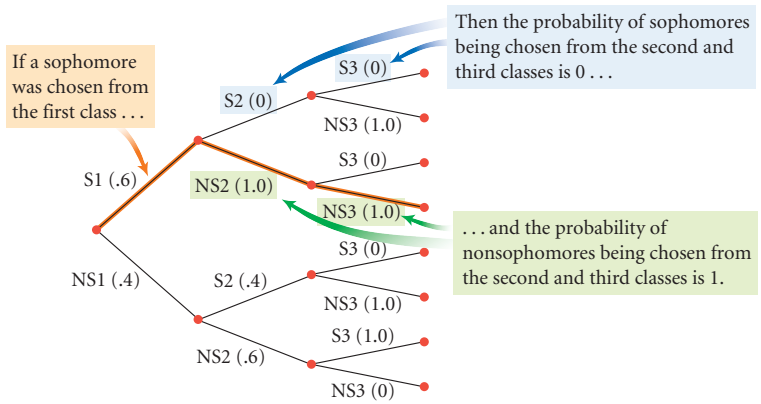
Tree diagrams will help you determine these probabilities.

a. The three highlighted paths represent the different outcomes that include a single sophomore. The first path has probability (.6)(.6)(.5), or .18, the second path has probability (.4)(.4)(.5), or .08, and the last path has probability (.4)(.6)(.5), or .12. The probability of one of these paths occurring is .18 + .08 + .12, or .38. So, 38% of the 8000 total paths contain exactly one sophomore.



b. If only one sophomore is allowed, then the probability that you are selected depends on what happened in the first class. If a sophomore was selected earlier, then you cannot be chosen. So, $P(\text{you}) = 0$. You have a chance only

if a sophomore was not selected earlier. In that case, $P(\text{you}) = \frac{1}{20} = .05$, because there are 20 students in your class. There is a .4 probability that a nonsophomore will be chosen in the first class, and then a .05 probability that you will be chosen in the second class. So, your probability of being chosen for the competition is (.4)(.05), or .02. You might suggest that Mr. Roark use a more fair method of selecting the team!



When the probability of an event depends on the occurrence of another event, the events are **dependent.** Independent and dependent events can be described using **conditional probability.** When events A and B are dependent, the probability of A occurring given that B occurred is different from the probability of A by itself. The probability of A given B is denoted with a vertical line:

$$P(A \mid B)$$

In Examples B and C, the probability of a sophomore in class 2 given a sophomore in class 1 would be written as $P(\text{S2} \mid \text{S1})$. The fact that events S1 and S2 were dependent in Example C means that $P(\text{S2} \mid \text{S1}) \neq P(\text{S2})$. In Example B, however, S1 and S2 were independent, so $P(\text{S2} \mid \text{S1}) = P(\text{S2})$.

You can use tree diagrams to break up dependent events into independent ones. In the tree diagram for Example C, part b, above, the probabilities of all branches other than the first branch are actually conditional probabilities. Events S1 and (S2 given S1) are independent, as are (S2 given S1) and (S3 given S1 and S2). So, you can use the multiplication rule to find the probabilities of the paths.

The Multiplication Rule (again)

If n_1, n_2, n_3, and so on, represent events along a path, then the probability that this sequence of events will occur can be found by multiplying the probabilities of the events.

$$P(n_1 \text{ and } n_2 \text{ and } n_3 \text{ and } \ldots) = P(n_1) \cdot P(n_2 \mid n_1) \cdot P(n_3 \mid (n_1 \text{ and } n_2)) \cdot \cdots$$

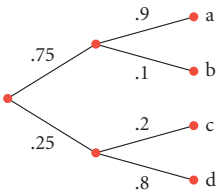
In the case of independent events, this statement is the same as the earlier statement of the multiplication rule, because $P(n_2 \mid n_1) = P(n_2)$, $P(n_3 \mid (n_1 \text{ and } n_2)) = P(n_3)$, and so on. In later lessons you'll see how to calculate the theoretical probabilities of other sorts of events.

EXERCISES

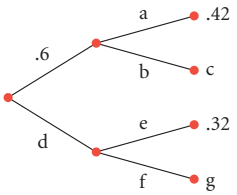
Practice Your Skills

1. Create a tree diagram showing the different outcomes if the cafeteria has three main entrée choices, two vegetable choices, and two dessert choices.

2. Find the probability of each path, a–d, in the tree diagram at right. What is the sum of the values of a, b, c, and d?



3. Find the probability of each path, a–g, in the tree diagram below.

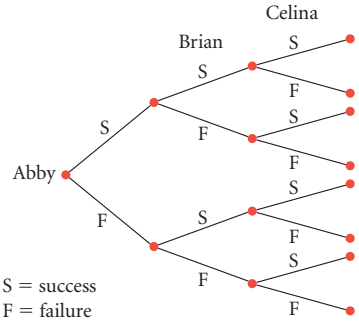


4. Three friends are auditioning for different parts in a comedy show. Each student has a 50% chance of success. Use the tree diagram at right to answer 4a–c.

a. Find the probability that all three students will be successful.

b. Find the probability that exactly two students will be successful.

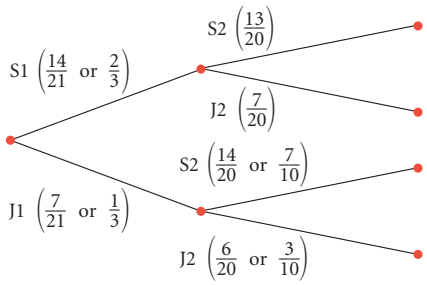
c. If you know that exactly two students have been successful, but do not know which pair, what is the probability that Celina was successful?



Keymath.com More Practice Your Skills

Reason and Apply

5. Explain the branch probabilities listed on this tree diagram, which models the outcomes of selecting two different students from a class of 7 juniors and 14 sophomores.



6. Use the diagram from Exercise 5 to answer each question.

a. Use the multiplication rule to find the probability of each path.

b. Are the paths equally likely? Explain.

c. What is the sum of the four answers in 6a?

7. A recipe calls for four ingredients: flour, baking powder, shortening, and milk (F, B, S, M). But there are no directions stating the order in which they should be combined. Chris has never followed a recipe like this before and has no idea which order is best, so he chooses the order at random.

a. How many different possible orders are there?

b. What is the probability that milk should be first?

c. What is the probability that the correct order includes flour first and shortening second?

d. What is the probability that the order is FBSM?

e. What is the probability that the order isn't FBSM?

f. What is the probability that flour and milk are next to each other?

8. Draw a tree diagram that pictures all possible equally likely outcomes if a coin is flipped as specified.

a. two times **b.** three times **c.** four times

9. How many different equally likely outcomes are possible if a coin is flipped as specified?

a. two times **b.** three times **c.** four times

d. five times **e.** ten times **f.** n times

10. You are totally unprepared for a true-false quiz, so you decide to guess randomly at the answers. There are four questions. Find the probabilities described in 10a–e.

a. P(none correct)

b. P(exactly one correct)

c. P(exactly two correct)

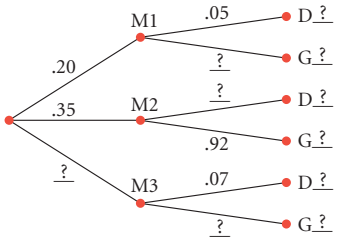
d. P(exactly three correct)

e. P(all four correct)

f. What should be the sum of the five probabilities in 10a–e?

g. If a passing grade means you get at least three correct answers, what is the probability that you passed the quiz?

11. **APPLICATION** The ratios of the number of phones manufactured at three sites, M1, M2, and M3, are 20%, 35%, and 45%, respectively. The diagram at right shows some of the ratios of the numbers of defective (D) and good (G) phones manufactured at each site. The top branch indicates a .20 probability that a phone made by this manufacturer was manufactured at site M1. The ratio of these phones that are defective is .05. Therefore, .95 of these phones are good. The probability that a randomly selected phone is both from site M1 and defective is (.20)(.05), or .01.



a. Copy the diagram and fill in the missing probabilities.

b. Find P(a phone from site M2 is defective).

c. Find P(a randomly chosen phone is defective).

d. Find P(a phone is manufactured at site M2 if you already know it is defective).

12. The Pistons and the Bulls are tied, and time has run out in the game. However, the Pistons have a player at the free throw line, and he has two shots to make. He generally makes 83% of the free throw shots he attempts. The shots are independent events, so each one has the same probability. Find these probabilities:

a. P(he misses both shots)

b. P(he makes at least one of the shots)

c. P(he makes both shots)

d. P(the Pistons win the game)

13. What is the probability that there are exactly two girls in a family with four children? Assume that girls and boys are equally likely.

14. The table at right gives numbers of students in several categories. Are the events "10th grade" and "Female" dependent or independent? Explain your reasoning.

	Male	Female	Total
10th grade	263	249	512
11th grade	243	234	477
12th grade	220	215	435
Total	726	698	1424

15. **APPLICATION** In 1963, the U.S. Postal Service introduced the ZIP code to help process mail more efficiently.

a. A ZIP code contains five digits, 0–9. How many possible ZIP codes are there?

b. In 1983, the U.S. Postal Service introduced ZIP + 4. The extra four digits at the end of the ZIP code help pinpoint the destination of a parcel with greater accuracy and efficiency. How many possible ZIP + 4 codes are there?

Postal workers sort through bulk mail.

The Canadian postal service uses a six-character mailing code of the form

letter, digit, letter, digit, letter, digit

c. How many possible Canadian postal codes are there if no restrictions are placed on the letters and digits?

d. In Canadian postal codes, the letters D, F, I, O, Q, and U are never used and the letters W and Z are not used as the first characters. How many possible postal codes are there now?

16. Braille is a form of writing for sight-impaired people. Each Braille character consists of a cell containing six positions that can have a raised dot or not have one. How many different Braille characters are possible?

History CONNECTION

At age 12, French innovator and teacher Louis Braille (1809–1852) invented a code that enables sight-impaired people to read and write. He got the idea from a former soldier who used a code consisting of up to 12 raised dots that allowed soldiers to share information on a dark battlefield without having to speak. Braille modified the number of dots to 6, and added symbols for math and music. In 1829, he published the first book in Braille.

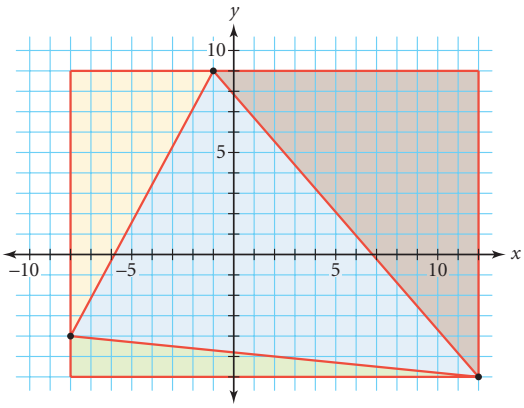
American author and lecturer Helen Keller (1880–1968) was both blind and deaf. This photo shows her with a Braille chart.

Review

17. Write each expression in the form $a + bi$.

a. $(2 + 4i) - (5 + 2i)$ **b.** $(2 + 4i)(5 + 2i)$ **c.** $\dfrac{2 + 4i}{5 + 2i}$

18. What is the probability that a randomly selected point within the rectangle at right is in the orange region? The blue region?



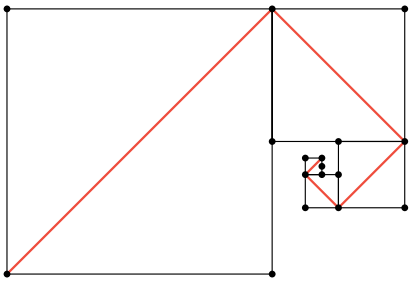
19. A sample of 230 students is categorized as shown.

	Male	Female
Junior	60	50
Senior	70	50

a. What is the probability that a junior is female?

b. What is the probability that a student is a senior?

20. The side of the largest square in the diagram at right is 4. Each new square has side length equal to half of the previous one. If the pattern continues infinitely, what is the long-run length of the spiral made by the diagonals?



EXPLORATION

The Law of Large Numbers

You have calculated both experimental and theoretical probabilities of events. In some cases, these probabilities are close. But when might they be different? And how can you predict the outcome of a single random process? In the activity you will explore ways to generalize the probability of an event.

Activity

A Repeat Performance

In this activity you will use a random number generator to simulate the roll of a die. You will then explore the effects of different numbers of trials.

Step 1 Open a new Fathom document. Insert a new case table. Enter the column heading, or attribute, Roll.

Step 2 Click on the attribute Roll. Choose **Edit Formula** from the Edit menu. To simulate the roll of a standard six-sided die, enter **randomPick(1, 2, 3, 4, 5, 6).** This command tells the computer to pick randomly any of the integers 1 through 6.

Step 3 From the Data menu, choose **New Cases.** Enter 10 as the number of new cases. Your case table will fill with 10 randomly chosen integers between 1 and 6.

Predict the shape of the histogram of your 10 data items. Then make a histogram and compare it to your prediction. To make a histogram, open a new graph. Drag the attribute Roll to the *x*-axis of your graph. Then change the graph type to histogram.

Step 4 What is the mode of your data? What are the maximum and minimum number of occurrences of outcomes? What is the difference between these values (the range)?

Step 5 Choose **Rerandomize** from the Analyze menu to generate another set of cases. Does the random generator seem to favor one number over the others? Does the **randomPick** command seem to simulate a fair die? How can you check?

Step 6 Predict the shape of a histogram for 100 cases. How do you think it would compare to a histogram for 10 items?

Step 7 Choose **New Cases** from the Data menu again and add 90 new cases. Your histogram should update automatically. Compare it to your prediction and to the histogram for 10 cases.

Step 8 What are the mode and range of possible outcomes? Rerandomize to generate another set of cases. Does the random generator seem to favor one number over the others?

Step 9 | Predict the histogram for 1000 cases. Then make a histogram and repeat Step 8.

A die that is not fair is called "loaded." A loaded die is designed to roll certain numbers more often.

Step 10 | Create a loaded die by using **randomPick** and any selection of numbers between 1 and 6. Let a classmate guess how the die is loaded by observing only 10 cases at a time.

Step 11 | Now repeat Step 10, but allow your classmate to use 1000 cases at a time.

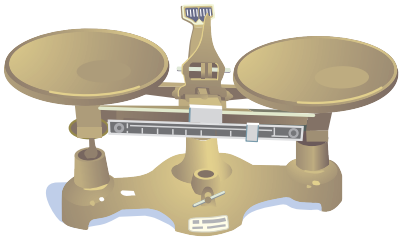
Questions

1. How does increasing the number of cases affect the shape of the histogram?
2. How does increasing the number of cases affect the range of data?
3. What strategies did you use to judge whether a die was loaded? Suppose you suspect that a die is loaded. How can you show that it is?
4. The Law of Large Numbers states that if an experiment is repeated many times, the experimental probability will get closer and closer to the theoretical probability. How does the Law of Large Numbers apply to this activity?

IMPROVING YOUR REASONING SKILLS

The Fake Coin

Angelina's grandfather gives her a collection of rare coins. It includes a box with 81 identical-looking coins. He tells her that one coin is fake because it is slightly lighter than all the others, but he does not remember which one it is. Angelina has a pan balance. What is the fewest number of balancings that she needs to make to be sure she will find the fake coin?



LESSON 12.3

Keymath.com Links to Resources

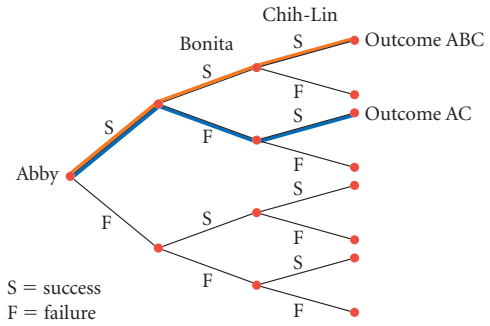
Mutually Exclusive Events and Venn Diagrams

Of course there is no formula for success except perhaps an unconditional acceptance of life and what it brings.

ARTHUR RUBENSTEIN

Two outcomes or events that cannot both happen are **mutually exclusive.** You already worked with theoretical probabilities and mutually exclusive events when you added probabilities of different paths.

For example, this tree diagram represents the possibilities of success in auditions by Abby, Bonita, and Chih-Lin. One path from left to right represents the outcome that Abby and Chih-Lin are successful but Bonita is not (outcome AC). Another outcome is success by all three (outcome ABC). These two outcomes cannot both take place, so they're mutually exclusive.



Suppose that each student has a .5 probability of success. Then the probability of any single path in the tree is (.5)(.5)(.5) = .125. So, the probability that either AC or ABC occurs is the sum of the probabilities on two particular paths, .125 + .125, or .25.

Untitled (1988) by British-American artist Judy Pfaff (b 1946) shows overlapping circles similar to Venn diagrams.

Just as tree diagrams allow breaking down sequences of dependent events into sequences of independent events, there's a tool for breaking down non–mutually exclusive events into mutually exclusive events. This tool is the Venn diagram, consisting of overlapping circles.

EXAMPLE A

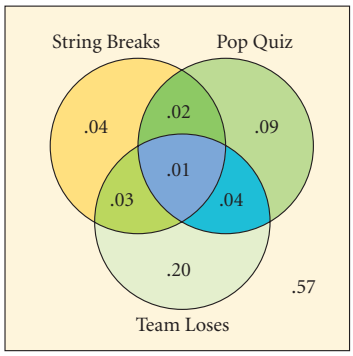
Melissa has been keeping a record of probabilities of events involving

i. Her guitar string breaking during orchestra rehearsal (Event B).

ii. A pop quiz in math (Event Q).

iii. Her team losing in gym class (Event L).

Although the three events are not mutually exclusive, they can be broken into eight mutually exclusive events. These events and their probabilities are shown in the Venn diagram.



a. What is the meaning of the region labeled .01?

b. What is the meaning of the region labeled .03?

c. What is the probability of a pop quiz, $P(\text{Q})$, today?

d. Find the probability of a pretty good day, $P(\text{not B and not Q and not L})$. This means no string breaks and no quiz and no loss.

▶ Solution

The meaning of each region is determined by the circles that contain it.

a. The region labeled .01 represents the probability of a really bad day. In this intersection of all three circles Melissa's string breaks, she gets a pop quiz, and her team loses.

b. The region labeled .03 represents the probability that Melissa's string will break and her team will lose, but there will be no pop quiz in math.

c. You can find the probability of a pop quiz by adding the four areas that are part of the pop-quiz circle: $.02 + .09 + .01 + .04 = .16$.

d. The probability of a pretty good day, $P(\text{not B and not Q and not L})$ is pictured by the region outside the circles, and is .57.

In general, the probability that one of a set of of mutually exclusive events will occur is the sum of the probabilities of the individual events.

The Addition Rule for Mutually Exclusive Events

If n_1, n_2, n_3, and so on represent mutually exclusive events, then the probability that any event in this collection of mutually exclusive events will occur is the sum of the probabilities of the individual events.

$$P(n_1 \text{ or } n_2 \text{ or } n_3 \text{ or } \ldots) = P(n_1) + P(n_2) + P(n_3) + \cdots$$

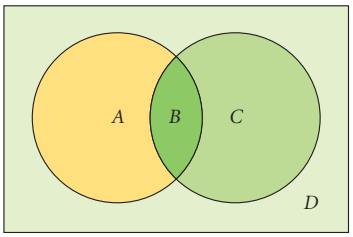
But what if you don't know all the probabilities that Melissa knew? In the investigation you'll discover one way to figure out probabilities of mutually exclusive events when you know probabilities of non–mutually exclusive events.

Investigation
Addition Rule

Of the 100 students in 12th grade, 70 are enrolled in mathematics, 50 are in science, 30 are in both subjects, and 10 are in neither subject.

Step 1 "A student takes mathematics" and "a student takes science" are two events. Are these events mutually exclusive? Explain.

Step 2 Complete a Venn diagram, similar to the one below, that shows enrollments in mathematics and science courses.



Step 3 Use the numbers of students in your Venn diagram to calculate probabilities.

Step 4 Explain why the probability that a randomly chosen student takes mathematics or science, $P(\text{M or S})$, does not equal $P(\text{M}) + P(\text{S})$.

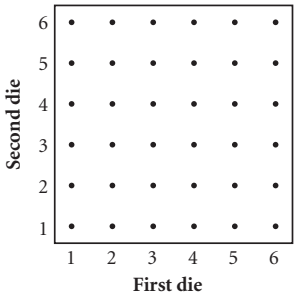
Step 5 Create a formula for calculating $P(\text{M or S})$ that includes the expressions $P(\text{M})$, $P(\text{S})$, and $P(\text{M and S})$.

Step 6 Suppose two dice are tossed. Draw a Venn diagram to represent the events

A = "sum is 7"

B = "each die > 2"

Find the probabilities in parts a–e by counting dots:



a. $P(\text{A})$

b. $P(\text{B})$

c. $P(\text{A and B})$

d. $P(\text{A or B})$

e. $P(\text{not A and not B})$

f. Find $P(\text{A or B})$ by using a rule or formula similar to your response in Step 5.

Step 7 Complete the statement: For any two events A and B, $P(\text{A or B}) = \underline{?}$.

A general form of the addition rule allows you to find the probability of an "or" statement even when two events are not mutually exclusive.

The General Addition Rule

If n_1 and n_2 represent event 1 and event 2, then the probability that at least one of the events will occur can be found by adding the probabilities of the events and subtracting the probability that both will occur.

$$P(n_1 \text{ or } n_2) = P(n_1) + P(n_2) - P(n_1 \text{ and } n_2)$$

You might wonder whether independent events and mutually exclusive events are the same. Independent events don't affect the probabilities of each other. Mutually exclusive events affect each other dramatically: If one occurs, the probability of the other occurring is 0.

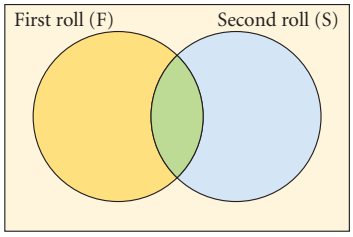
But there is a connection between independent and mutually exclusive events. In calculating probabilities of non–mutually exclusive events, you use the probability that they both will occur. In the case of independent events, you know this probability.

EXAMPLE B

The probability that a rolled die comes up 3 or 6 is $\frac{1}{3}$. What's the probability that a die will come up 3 or 6 on the first and/or second roll?

▶ Solution

Let F represent getting a 3 or 6 on the first roll, regardless of what happens on the second roll. Let S represent getting a 3 or 6 on the second roll, regardless of what happens on the first roll. Notice that both F and S include the possibility of getting a 3 or 6 on both rolls, as shown by the overlap in the Venn diagram. To account for the overlap, the general addition rule subtracts $P(\text{F and S})$ once.



$P(\text{F or S}) = P(\text{F}) + P(\text{S}) - P(\text{F and S})$ — The general addition rule.

$P(\text{F or S}) = P(\text{F}) + P(\text{S}) - P(\text{F}) \cdot P(\text{S})$ — F and S are independent, so $P(\text{F and S}) = P(\text{F}) \cdot P(\text{S})$.

$P(\text{F or S}) = \frac{1}{3} + \frac{1}{3} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$ — Substitute the probability.

$P(\text{F or S}) = \frac{5}{9}$. — Multiply and add.

The probability of getting a 3 or 6 on the first and/or second roll is $\frac{5}{9}$.

When two events are mutually exclusive and make up all possible outcomes, they are referred to as **complements.** The complement of "a 1 or a 3 on the first roll" is "not a 1 and not a 3 on the first roll," an outcome that is represented by the regions outside the F circle. Because $P(\text{F})$ is $\frac{1}{3}$, the probability of the complement, $P(\text{not F})$, is $1 - \frac{1}{3}$, or $\frac{2}{3}$.

EXAMPLE C

Every student in the school music program is backstage, and no other students are present. Use O to represent the event that a student is in the orchestra, C to represent the event that a student is in the choir, and B to represent the event that a student is in the band. A reporter who approaches a student at random backstage knows these probabilities:

i. $P(\text{B or C}) = .8$

ii. $P(\text{not O}) = .6$

iii. $P(\text{C and not O and not B}) = .1$

iv. O and C are independent

v. O and B are mutually exclusive

a. Turn each of these statements into a statement about percentage in plain English.

b. Create a Venn diagram of probabilities describing this situation.

▶ Solution

a. Convert each probablility statement to a percentage statement.

i. $P(\text{B or C}) = .8$ means that 80% of the students are in band or in choir.

ii. $P(\text{not O}) = .6$ means that 60% of the students are not in orchestra.

iii. $P(\text{C and not O and not B}) = .1$ means that 10% of the students are in choir only.

iv. "O and C are independent" means that the percentage of students in choir is the same as the percentage of orchestra students in choir. Being in orchestra does not make a student any more or less likely to be in choir, and vice versa.

v. "O and B are mutually exclusive" means that there are no students in both orchestra and band.

b. Start with a general Venn diagram showing the overlap of 3 events.

Because every student backstage is in the music program, the probability $P(\text{not C and not O and not B})$ is 0. So, the region Z in the Venn diagram below has a probability of 0.

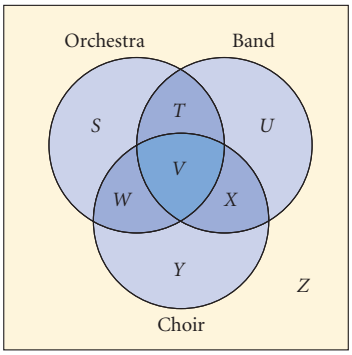
Because O and B are mutually exclusive, $T = 0$ and $V = 0$.

$P(\text{C and not O and not B}) = .1$ means that $Y = .1$.

$P(\text{B or C}) = .8$ means that $U + W + X + Y = .8$.

$P(\text{not O}) = .6$ means that $U + X + Y = .6$.

The difference in the last two statements, $(U + W + X + Y) - (U + X + Y)$, is W, so $W = .8 - .6 = .2$.



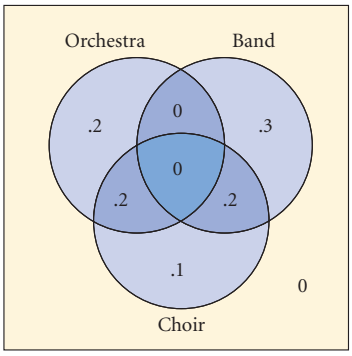
$P(\text{not O}) = .6$ means $P(\text{O}) = .4$. Therefore, $S + W + T + V$ equals .4. Substituting the known values of W, T, and V, gives $S + .2 = .4$, so $S = .2$.

Because O and C are independent,

$$P(\text{O}) \cdot P(\text{C}) = P(\text{O and C})$$
$$(S + W)(W + X + Y) = W$$
$$.4(.2 + .1 + X) = .2$$
$$.3 + X = \frac{.2}{.4} = .5$$
$$X = .2$$

Finally returning to $U + X + Y = .6$, substitute the values of X and Y to get $U = .3$.

Write the probabilities in your Venn diagram.

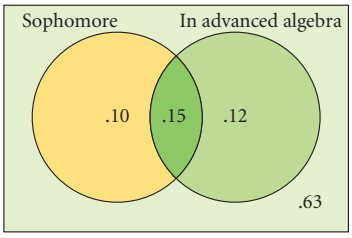


You should take the time to check that the sum of the probabilities in all the areas is 1. Solving this kind of puzzle exercises your understanding of probabilities and the properties of *and, or, independent,* and *mutually exclusive.*

EXERCISES

Practice Your Skills

Exercises 1–4 refer to the diagram at right. Let S represent the event that a student is a sophomore, and A represent the event that a student takes advanced algebra.



1. Describe the events represented by each of the four regions.

2. Find the probability of each event.

a. $P(\text{S})$ **b.** $P(\text{A and not S})$

c. $P(\text{S} \mid \text{A})$ **d.** $P(\text{S or A})$

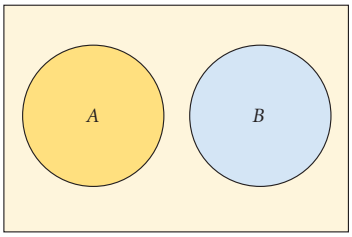
3. Suppose the diagram refers to a high school with 500 students. Change each probability into a frequency (number of students).

4. Are the two events, S and A, independent? Show mathematically that you are correct.

5. Events A and B are pictured in the Venn diagram at right.

a. Are the two events mutually exclusive? Explain.

b. Are the two events independent? Assume $P(A) \neq 0$ and $P(B) \neq 0$. Explain.



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Reason and Apply

6. Of the 420 students at Middletown High School, 126 study French and 275 study music. Twenty percent of the music students take French.

a. Create a Venn diagram of this situation.

b. What percentage of the students take both French and music?

c. How many students take neither French nor music?

7. Two events, A and B, have probabilities $P(A) = .2$, $P(B) = .4$, $P(A \mid B) = .2$.

a. Create a Venn diagram of this situation.

b. Find the value of each probability indicated.

i. $P(A \text{ and } B)$

ii. $P(\text{not } B)$

iii. $P(\text{not } (A \text{ or } B))$

8. If $P(A) = .4$ and $P(B) = .5$, what range of values is possible for $P(A \text{ and } B)$? What range of values is possible for $P(A \text{ or } B)$? Use a Venn diagram to help explain how each range is possible.

9. Assume that the diagram for Exercises 1–4 refers to a high school with 800 students. Draw a new diagram showing the probabilities if 20 sophomores moved from geometry to advanced algebra.

10. At right are two color wheels. Figure A represents the mixing of light, and Figure B represents the mixing of pigments.

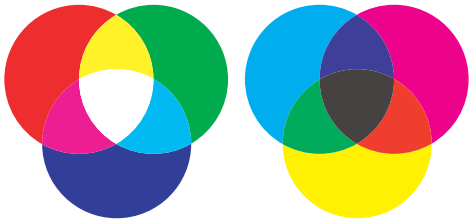


Figure A Figure B

Using Figure A, what color is produced when equal amounts of

a. Red and green light are mixed?

b. Blue and green light are mixed?

c. Red, green, and blue light are mixed?

Using Figure B, what color is produced when equal amounts of

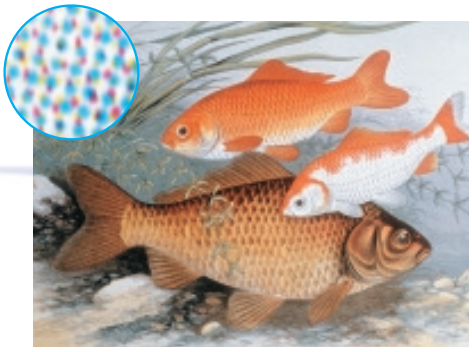
d. Magenta and cyan pigments are mixed?

e. Yellow and cyan pigments are mixed?

f. Magenta, cyan, and yellow pigments are mixed?

Science CONNECTION

The three primary colors of light are red, green, and blue, each having its own range of wavelengths. When these waves reach our eyes, we see the color associated with the reflected wave. These colors are also used to project images on TV screens and computer monitors and in lighting performances on stage. In the mixing of colors involving paint, ink, or dyes, the primary colors are cyan (greenish-blue), magenta (purplish-red), and yellow. Mixing other pigment colors cannot duplicate these three colors. Pigment color mixing is important in the textile industry as well as in art, design, and printing.



This magnified image shows how the color in a printed photograph appears on paper. The colors are combinations of cyan, magenta, yellow, and black. These four colors can be blended in varying amounts to create any color.

11. Kendra needs help on her math homework and decides to call one of her friends—Amber, Bob, or Carol. Kendra knows that Amber is on the phone 30% of the time, Bob is on the phone 20% of the time, and Carol is on the phone 25% of the time.

a. If the three friends' phone usage is independent, make a Venn diagram of the situation.

b. What is the probability that all three of her friends will be on the phone when she calls?

c. What is the probability that none of her friends will be on the phone when she calls?

Review

12. The registered voters represented in the table have been interviewed and rated. Assume that this sample is representative of the voting public in a particular town. Find each probability.

	Liberal	Conservative
Age under 30	210	145
Age 30–45	235	220
Age over 45	280	410

a. P(a randomly chosen voter is over 45 yr old and liberal)

b. P(a randomly chosen voter is conservative)

c. P(a randomly chosen voter is conservative if under 30 yr old)

d. P(a randomly chosen voter is under 30 yr old if conservative)

13. The most recent test scores in a chemistry class were

{74, 71, 87, 89, 73, 82, 55, 78, 80, 83, 72}

What was the average (mean) score?

14. If an unfair coin has $P(\text{H}) = \frac{2}{5}$ and $P(\text{T}) = \frac{3}{5}$, then what is the probability that six flips come up with H, T, T, T, H, T in exactly this order?

15. Rewrite each expression in the form $a\sqrt{b}$, such that b contains no factors that are perfect squares.

a. $\sqrt{18}$ **b.** $\sqrt{54}$ **c.** $\sqrt{60x^3y^5}$

16. Port Charles and Turner Lake are 860 km apart. At 6:15 A.M., Patrick and Ben start driving from Port Charles to Turner Lake at an average speed of 80 km/h. At 9:30 A.M., Carl and Louis decide to meet them. They leave Turner Lake and drive at 95 km/h toward Port Charles. When and where do the two parties meet?

LESSON

12.4

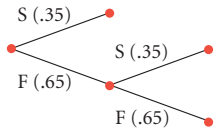
Random Variables and Expected Value

The real measure of your wealth is how much you'd be worth if you lost all your money.

ANONYMOUS

Imagine that you are sitting near the rapids on the bank of a rushing river. Salmon are attempting to swim upstream. They must jump out of the water to pass the rapids. While sitting on the bank, you observe 100 salmon jumps and, of those, 35 are successful. Having no other information, you can estimate that the probability of success is 35% for each jump.

What is the probability that a salmon will make it on its second attempt? Note that this situation requires that two conditions be met: that the salmon fails on the first jump, and that it succeeds on the second jump. In the diagram, you see that this probability is (.65)(.35) = .2275, or about 23%. To determine the probability that the salmon makes it on the first or second jump, you sum the probabilities of the two mutually exclusive events: making it on the first jump and making it on the second jump. The sum is .35 + .2275 = .5775, or about 58%.



Sockeye salmon swim upstream to spawn.

Probabilities of success such as these are often used to predict the number of independent trials required before the first success (or failure) is achieved. The salmon situation gave experimental probabilities. In the investigation you'll explore theoretical probabilities associated with dice.

Investigation

"Dieing" for a Four

Each person will need a single die. [▶ See **Calculator Note 1L** to simulate rolling a die if you don't have one.◀] Imagine that you're about to play a board game in which you must roll a 4 on your die before taking your first turn.

Step 1 Record the number of rolls it takes for you to get a 4. Repeat this a total of ten times. Combine your results with those of your group and find the mean of all values. Then find the mean of all the group results in the class.

Step 2 Based on this experiment, how many rolls would you expect to make on average before a 4 comes up?

Step 3 To calculate the result more theoretically, imagine a "perfect" sequence of rolls, with the results 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, and so on. On average, how many rolls do you need after each 4 to get the next 4?

Step 4 Another theoretical approach uses the fact that the probability of success is $\frac{1}{6}$. Calculate the probability of rolling the first 4 on the first roll, the first 4 on the second roll, the first 4 on the third roll, and the first 4 on the fourth roll. (A tree diagram might help you do the calculations.)

Step 5 Find a formula for the probability of rolling the first 4 on the nth roll.

Step 6 Create a calculator list with the numbers 1 through 100. [▶ See **Calculator Note 12B** for a quick way to enter sequences into lists.◀] Use your formula from Step 5 to make a second list of the probabilities of rolling the first 4 on the first roll, the second roll, the third roll, and so on, up to the 100th roll. Create a third list that is the product of these two lists. Calculate the sum of this third list. [▶ See **Calculator Note 2B.**◀]

Step 7 How close is the sum you found in Step 6 to your estimates in Steps 2 and 3?

The average value you found in Steps 2, 3, and 6 is called the **expected value.** It is also known as the long-run value, or mean value.

But it's the expected value of what? A numerical quantity whose value depends on the outcome of a chance experiment is called a **random variable.** In the investigation the random variable gave the number of rolls before getting a 4 on a die. You found the expected value of that random variable. The number of jumps a salmon makes before succeeding is another example of a random variable. Both of these variables are **discrete random variables** because their values are integers. They're also called **geometric random variables** because they are a count of the number of independent trials before something happens (success or failure). You will discover why these variables can be called *geometric* in the exercise set for this lesson.

The following example shows discrete random variables that are not geometric.

EXAMPLE A

When two fair dice are rolled, the sum of the results varies.

a. What are the possible values of the random variable, and what probabilities are associated with those values?

b. What is the expected value of this random variable?

▶ Solution

The random variable x has as its values all possible sums of the two dice.

a. The values of x are the integers such that $2 \leq x \leq 12$. The table shows each value and its probability, computed by counting numbers of outcomes.

x	2	3	4	5	6	7	8	9	10	11	12
Probability $P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b. The expected value of x is the theoretical average you'd expect to have after many rolls of the dice. Your intuition may tell you that the expected value is 7. One way of finding this weighted average is to imagine 36 "perfect" rolls, so every possible outcome occurs exactly once. The mean of the values is

$$\frac{2+3+3+4+4+4+\cdots+11+11+12}{36} = 7$$

If you distribute the denominator over the terms in the numerator, and group like outcomes, you get an equivalent expression that uses the probabilities:

$$\frac{1}{36}\cdot 2+\frac{2}{36}\cdot 3+\frac{3}{36}\cdot 4+\frac{4}{36}\cdot 5+\frac{5}{36}\cdot 6+\frac{6}{36}\cdot 7+\frac{5}{36}\cdot 8+$$
$$\frac{4}{36}\cdot 9+\frac{3}{36}\cdot 10+\frac{2}{36}\cdot 11+\frac{1}{36}\cdot 12 = 7$$

Note that each term in this expression is equivalent to the product of a value of x and the corresponding probability, $P(x)$, in the table on page 688.

Expected Value

The expected value of a random variable is an average value found by multiplying the value of each event by its probability and then summing all of the products. If $P(x)$ is the probability of an event with a value of x, then the expected value, $E(x)$, is given by the formula

$$E(x) = \sum xP(x)$$

Even if a random variable is discrete, its expected value may not be an integer.

EXAMPLE B

When Nate goes to visit his grandfather, his grandfather always gives him a piece of advice and a bill from his wallet. Grandpa closes his eyes and takes out one bill and gives it to Nate. On this visit he sends Nate to get his wallet. Nate peeks inside and sees 8 bills: 2 one-dollar bills, 3 five-dollar bills, 2 ten-dollar bills, and 1 twenty-dollar bill. What is the expected value of the bill Nate will receive?

▶ Solution

The random variable x takes on 4 possible values, and each has a known probability.

Outcome x	\$1	\$5	\$10	\$20	
Probability $P(x)$	.25	.375	.25	.125	**Sum**
Product $x \cdot P(x)$	0.25	1.875	2.5	2.5	7.125

The expected value is \$7.125.

The other approach, using division, gives the same result:

$$\frac{1 + 1 + 5 + 5 + 5 + 10 + 10 + 20}{8} = \frac{57}{8} = 7.125$$

Nate doesn't actually expect Grandpa to pull out a \$7.125 bill. But if Grandpa did the activity over and over again, with the same bills, the value of the bill would average \$7.125. The expected value applies to a single trial, but it's based on an average over many imagined trials.

In Example B, suppose Nate always had the same choice of bills and that he had to pay his grandfather \$7 for the privilege of receiving a bill. Over the long run, he'd make \$0.125 per trial. On the other hand, if Nate had to pay \$8 each time, his grandfather would make \$0.875 on average per trial. This is the principle behind the way that raffles and casinos make money. Sometimes gamblers win, but on average gamblers lose.

Consumer CONNECTION

Casinos make money because on average gamblers bet more than they win. The amount by which the casino is favored is called the "house edge," calculated by finding the ratio of the casino's expected earnings to the player's initial wager. There are two ways that a casino creates an edge. In some games, like Blackjack, the probability is simply higher that the dealer will win. In a game like Roulette, the casino pays a winner at a rate less than the actual odds of winning. Either way, the casino always profits in the long run.

The name Roulette is derived from the Old French word *roelete*, meaning little wheel.

You can use either the mean or the probability approach to find the expected value if the random variable takes on a finite number of values. But there's no theoretical limit to the number of times you might have to roll a die before getting a 4. If the random variable has infinitely many values, you must use the probability approach to find the expected value.

EXERCISES

Practice Your Skills

1. Which of these numbers comes from a discrete random variable? Explain.

a. the number of children who will be born to members of your class

b. the length of your pencil in inches

c. the number of pieces of mail in your mailbox today

2. Which of these numbers comes from a geometric random variable? Explain.
   a. the number of phone calls a telemarketer makes until she makes a sale
   b. the number of cats in the home of a cat owner
   c. the number of minutes until the radio plays your favorite song

3. You have learned that 8% of the students in your school are left-handed. Suppose you stop students at random and ask whether they are left-handed.
   a. What is the probability that the first left-handed person you find is on your third try?
   b. What is the probability that you will find exactly one left-handed person in three tries?

4. Suppose you are taking a multiple-choice test for fun. Each question has five choices (A–E). You roll a six-sided die and mark the answer according to the number on the die and leave the answer blank if the die is a 6. Each question is scored one point for a right answer, minus one-quarter point for a wrong answer, and no points for a question left blank.
   a. What is the expected value for each question?
   b. What is the expected value for a 30-question test?

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Reason and Apply

5. Sly asks Andy to play a game with him. They each roll a die. If the sum is greater than 7, Andy scores 5 points. If the sum is less than 8, Sly scores 4 points.
   a. Find a friend and play the game ten times. [▶ See **Calculator Note 1L** if you have no dice. ◀] Record the final score.
   b. What is the experimental probability that Andy will have a higher score after ten games?
   c. Draw a tree diagram of this situation showing the theoretical probabilities.
   d. If you consider this game from Andy's point of view, his winning value is $+5$ and his losing value is -4. What is the expected value of the game from his point of view?
   e. Suggest a different distribution of points that would favor neither player.

6. Suppose each box of a certain kind of cereal contains a card with one letter from the word CHAMPION. The letters have been equally distributed in the boxes. You win a prize when you send in all eight letters.
   a. Predict the number of boxes you would expect to buy to get all eight letters.
   b. Describe a method of modeling this problem using the random-number generator in your calculator.
   c. Use your method to simulate winning the prize. Do this five times. Record your results.
   d. What is the average number of boxes you will have to buy to win the prize?
   e. Combine your results with those of several classmates. What seems to be the overall average number of boxes needed to win the prize?

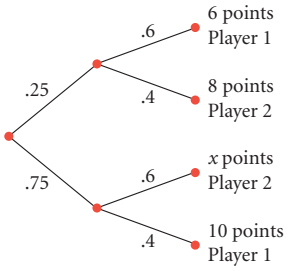
7. In a concert hall, 16% of seats are in section A, 24% are in section B, 32% are in section C, and 28% are in section D. Section A seats sell for \$35, section B for \$30, section C for \$25, and section D for \$15. You see a ticket stuck high in a tree.

a. What is the expected value of the ticket?

b. The markings on the ticket look like either section A or C. If this is true, then what is the probability that the ticket is for section C?

c. If the ticket is for section A or C, then what is the expected value of the ticket?

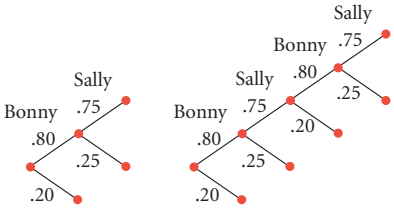
8. The tree diagram at right represents a game played by two players.



a. Find a value of x that gives approximately the same expected value for each player.

b. Design a game that could be described with this tree diagram. (You may use coins, dice, spinners, or some other device.) Explain the rules of the game and the scoring of points.

9. **APPLICATION** Bonny and Sally are playing in a tennis tournament. On average, Bonny makes 80% of her shots, whereas Sally makes 75% of hers. Bonny serves first. At right are tree diagrams depicting possible sequences of events for one and two volleys of the ball. (A volley is a single sequence of Bonny hitting the ball and Sally returning it.) Note that once a ball is missed, that branch ends.



a. What is the probability that Bonny will win the point after just one volley?

b. What is the probability that Bonny will win the point in exactly two volleys?

c. Make a tree diagram to represent three volleys. What is the probability that Bonny will win the point in exactly three volleys?

d. What is the probability that Bonny will win the point in at most three volleys?

e. What kind of sequence do the answers to 9a–c form? Explain.

f. What is the probability that Bonny will win the point in at most six volleys?

g. In the long run, what is the probability that Bonny will win the point?

Technology CONNECTION

In computer simulations of sporting events, software designers enter all the data and statistics they can find so that they can model the game as accurately as possible. In tennis, for instance, individual statistics on serves, backhand shots and forehand shots, and positions on the court are taken into account. When setting up a fantasy team or match, programmers base their formulas on actual probabilities, and even the best players or teams will lose some of the time.

American tennis players Venus and Serena Williams are shown here playing in the Women's Doubles at the 2003 Australian Open Tennis Championship in Melbourne, Australia.

10. A group of friends has a huge bag of red and blue candies, with approximately the same number of each color. Each of them picks out candies, one at a time, until he or she gets a red candy. The bag is then passed on to a friend.

a. Devise and describe a simulation for this problem. Use your simulation to approximate the long-run average number of candies each person pulled out in one turn.

b. What is the long-run average number of blue candies each person pulled out in one turn?

11. **APPLICATION** A quality-control engineer randomly selects five radios from an assembly line. Experience has shown that the probabilities of finding 0, 1, 2, 3, 4, or 5 defective radios are as shown in the table.

Number of defective radios x	Probability $P(x)$	$x \cdot P(x)$
0	.420	
1	.312	
2	.173	
3	.064	
4	.031	
5	.000	

a. What is the probability the engineer will find at least one defective radio in a random sample of five?

b. Make a copy of this table on your paper, and complete the entries for the column titled $x \cdot P(x)$.

c. Find the sum of the entries in the column titled $x \cdot P(x)$.

d. What is the real-world meaning of your answer in 11c?

12. What is the probability that a 6 will not appear until the eighth roll of a fair die?

Review

13. Find $P(E_1 \text{ or } E_2)$ if E_1 and E_2 are mutually exclusive and complementary.

14. Two varieties of flu spread through a school one winter. The probability that a student gets both varieties is .18. The probability that a student gets neither variety is .42. Having one variety of flu does not make a student more or less likely to get the other variety. What is the probability that a student gets exactly one of the flu varieties?

15. This table gives counts of different types of paperclips in Maricela's paperclip holder.

	Small	Large
Metal & oval	47	23
Plastic & oval	25	10
Metal & triangular	18	6
Plastic & triangular	10	5

a. Create a Venn diagram of the probabilities of picking each kind of clip if one is selected at random. Use metal, oval, and small as the categories for the three circles on your diagram.

b. Create a tree diagram of the outcomes and their probabilities. Use size on the first branch, shape on the second, and material on the third.

16. Solve $\left(\frac{1}{2}\right)^n \leq 10^{-5}$.

17. Suppose the probability of winning a single game of chance is .9. How many wins in a row would it take before the likelihood of such a string of wins would be less than 1%?

LESSON

12.5 Permutations and Probability

Keymath.com Links to Resources

Never say never. . . . Never is a long, undependable time, and life is too full of rich possibilities to have restrictions placed upon it.

GLORIA SWANSON

The numerator and denominator of a theoretical probability are numbers of possibilities. Sometimes those possibilities follow regular patterns that allow you to "count" them.

Investigation
Order and Arrange

You will need

- five different objects

Suppose you are working on a jigsaw puzzle. You get to a section of the puzzle where there are five spaces to be filled. Unfortunately, it looks as though they are all nearly the same shape and color. You grab a handful of the remaining pieces and start trying them. How many different ways could you try to fit the remaining pieces into the remaining spaces? If you try different random arrangements, how many arrangements are there to try?

Step 1 Investigate this problem using different objects to represent the puzzle pieces. Let n represent the number of objects and r the number of spaces, or slots, you need to fill. For example, $n = 3$ and $r = 2$ represents the number of different ways three pieces can fill two spaces.

In this case the order matters, so object A followed by object B is different from object B followed by object A. Copy and complete the table on the next page for different values of n and r.

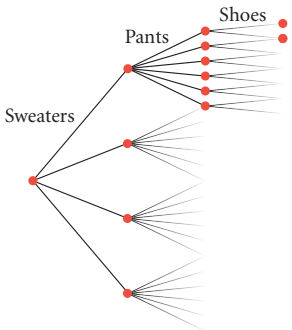
		Number of items, n				
		$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
Number of spaces, r	$r = 1$					
	$r = 2$			6		
	$r = 3$					
	$r = 4$					
	$r = 5$					

Step 2 Compare your results with those of your group. Describe any patterns you found in either the rows or columns of the table.

There are many patterns you might have seen in the investigation. One pattern can be described as a product of the numbers of outcomes that fill the slots. For example, for $n = 4$ and $r = 3$, you would make three slots because $r = 3$, and then fill the first slot with four choices and decrease by one choice for each slot as you go: 4 3 2. The product of those numbers is 24, the number in the third row, fourth column of the table you completed.

Why do you multiply the numbers in the slots? The problem might remind you of a familiar situation. How many different outfits—consisting of a sweater, pants, and shoes—could you wear if you were to select from four sweaters, six pairs of pants, and two pairs of shoes?

You can visualize a tree diagram with four choices of sweaters. For each of those sweaters, you can select six different pants. Each of those sweater and pants outfits can be matched with two pairs of shoes. (Actually drawing all of the paths would be difficult and messy.) Each different outfit is represented by a path of three segments representing a sweater *and* a pair of pants *and* a pair of shoes. The paths represent all the possible outcomes, or the different ways in which the entire sequence of choices can be made.



How many outfits are there? From the tree you're imagining, you can see that the total number of outfits with four choices, then six choices, and then two choices can be found by multiplying $4 \cdot 6 \cdot 2$. You are using the **counting principle.**

Counting Principle

Suppose there are n_1 ways to make a choice, and for each of these there are n_2 ways to make a second choice, and for each of these there are n_3 ways to make a third choice, and so on. The product $n_1 \cdot n_2 \cdot n_3 \cdot \cdots$ is the number of possible outcomes.

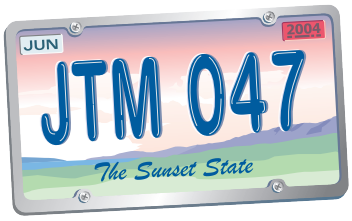
The counting principle provides a quick method for calculating numbers of outcomes, using multiplication. Rather than memorizing the formula, you can look for patterns, and sketch or visualize a representative tree diagram.

EXAMPLE A

Suppose a set of license plates has any three letters from the alphabet, followed by any three digits.

a. How many different license plates are possible?

b. What is the probability that a license plate has no repeated letters or numbers?



▶ Solution

To better understand the problem, fill in some slots.

a. There are 26 possible letters for each of the first three slots and 10 possible digits for each of the last three slots. Remember that letters and digits can repeat. Using the counting principle, you can multiply the number of possibilities:

$$\underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 17{,}576{,}000$$

There are 17,576,000 possible license plates.

b. Once the first letter is chosen, there are only 25 ways of choosing the second letter to avoid repetition. This pattern continues for the third letter and for the digits. Filling slots gives the product $\underline{26} \cdot \underline{25} \cdot \underline{24} \cdot \underline{10} \cdot \underline{9} \cdot \underline{8} = 11{,}232{,}000$. The probability that a license plate will be one of these arrangements is this number divided by the total number of possible outcomes:

$$\frac{11232000}{17576000} \approx .639$$

This means that about 63.9% of license plates would not have repeated letters or numbers.

This building is decorated with Colorado license plates.

When the objects cannot be used more than once, the number of possibilities decreases at each step. These are called "arrangements without replacement." In other words, once an item is chosen, that same item cannot be used again in the same arrangement.

An arrangement of some or all of the objects from a set, without replacement, is called a **permutation.** The notation ${}_nP_r$ is read "the number of permutations of n things chosen r at a time." As in the investigation and part b of Example A, you can calculate ${}_nP_r$ by multiplying $n(n-1)(n-2)(n-3)\cdots(n-r+1)$. [▶ See **Calculator Note 12C** to learn how to compute ${}_nP_r$ on a calculator.◀]

EXAMPLE B

Seven flute players are performing in an ensemble.

a. The names of all seven players are listed in the program in random order. What is the probability that the names are in alphabetical order?

b. After the performance, the players are backstage. There is a bench with only room for four to sit. How many possible seating arrangements are there?

c. What is the probability that the group of four players is sitting in alphabetical order?

▶ Solution

You can find numbers of permutations by filling in slots.

a. There are seven choices for the first name on the list, six choices remaining for the second name, five for the third name, and so on.

$${}_7P_7 = \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 5040$$

Only one of these arrangements is in alphabetical order. The probability is $\frac{1}{5040}$, or approximately .0002.

b. In this case there are only four slots to fill. There are seven choices for the first seat, six choices remaining for the second seat, five for the third seat, and four for the fourth seat.

$${}_7P_4 = \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} = 840$$

c. With each arrangement of 4, there is only one correct order:

$${}_4P_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

There are 24 ways to arrange each grouping of 4 players, so the probability is $\frac{1}{24}$, or approximately .04167.

Notice that the answer to part c does not depend on the answer to part b.

When $r = n$, as in parts a and c of Example B, you can see that ${}_nP_r$ equals the product of integers from n all the way down to 1. A product like this is called a **factorial** and is written with an exclamation point. For example, 7 factorial, or 7!, is $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, or 5040. [▶ See **Calculator Note 12D** to learn how to evaluate factorials on a calculator.◀]

Look again at Example B. You can write the solution to part a as

$${}_7P_7 = 7! = 5040$$

For part b, you need only the product of integers from 7 down to 4. You can write this as $\frac{7!}{3!}$.

$$\frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$$

You can use this idea to write a formula for any number of permutations, ${}_nP_r$. As you saw above, you can write the number as $\frac{n!}{(n-r)!}$.

To avoid division by 0 when $r = n$, we define 0! as equal to 1. So, when r and n are equal,

$${}_nP_r = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Permutations

A **permutation** is an arrangement of some or all of the objects from a set, without replacement.

The number of permutations of n objects chosen r at a time ($r \leq n$) is

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Verify that the formula above gives you the same values you found in the investigation.

In this lesson you used tree diagrams, the counting principle, your calculator, and perhaps other ways to count permutations. As you do the exercises, consider which strategy is best for each particular problem.

EXERCISES

Practice Your Skills

1. Screamers Ice Cream Parlor sells a triple-scoop cone. Which of these situations are permutations? For those that are not, tell why.

a. the different cones if all three scoops are different flavors and vanilla, then lemon, then mint is considered different from vanilla, then mint, then lemon

b. the different cones if all three scoops are different flavors and vanilla, then lemon, then mint is considered the same as vanilla, then mint, then lemon

c. the different cones if you can repeat a flavor two or three times and vanilla, then lemon, then lemon is different from lemon, then vanilla, then lemon

d. the different cones if you can repeat a flavor two or three times and vanilla, then lemon, then lemon is the same as lemon, then vanilla, then lemon

2. Evaluate the factorial expressions. (Some answers will be in terms of n.)

a. $\frac{12!}{11!}$ **b.** $\frac{7!}{6!}$ **c.** $\frac{(n+1)!}{n!}$

d. $\frac{n!}{(n-1)!}$ **e.** $\frac{120!}{118!}$ **f.** $\frac{n!}{(n-2)!}$

g. Find n if $\frac{(n+1)!}{n!} = 15$.

3. Evaluate each expression. (Some answers will be in terms of n.)

a. ${}_7P_3$ **b.** ${}_7P_6$ **c.** ${}_{n+2}P_n$ **d.** ${}_nP_{n-2}$

4. Consider making a four-digit I.D. number using the digits 3, 5, 8, and 0.

a. How many I.D. numbers can be formed using each digit once?

b. How many can be formed using each digit once and not using 0 first?

c. How many can be formed if repetition is allowed and any digit can be first?

d. How many can be formed if repetition is allowed but 0 is not used first?

5. **APPLICATION** A combination lock has four dials. On each dial are the digits 0 to 9.

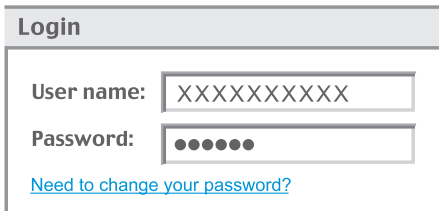
a. Suppose you forget the correct combination to open the lock. How many combinations do you have to try? If it takes 10 s to enter each combination, how long will it take you to try every possibility?

b. Suppose you replace your lock with one that has five dials, each with the digits 0 to 9. How many combinations are possible? If it still takes 10 s to enter each combination, how long will it take to try every possibility?

c. For a lock to be secure, it has to be difficult for someone to guess the correct combination. How many times as secure as a 4-dial lock is a 5-dial lock?

Consumer CONNECTION

Gaining access to personal bank, telephone, or e-mail accounts often requires using a password, usually a minimum of 4 digits and/or letters. Most businesses recommend that consumers change their passwords frequently and choose a password with more than six characters, in combinations of digits and upper- and lowercase letters, to increase the safety of their accounts. In addition, password programs are designed to automatically shut down after a certain number of consecutive incorrect passwords have been entered.



Keymath.com More Practice Your Skills

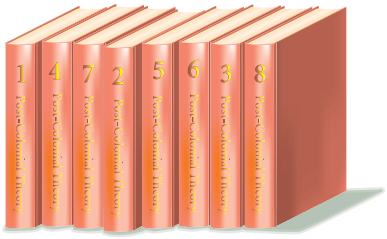
Reason and Apply

6. For what value(s) of n and r does ${}_nP_r = 720$? Is there more than one answer?

7. How many factors are in the expression $n(n-1)(n-2)(n-3)\cdots(n-r+1)$?

8. Suppose each student in a school is assigned one locker. How many ways can three new students be assigned to five available lockers?

9. An eight-volume set of reference books is kept on a shelf. The books are used frequently and put back in random order.



a. How many ways can the eight books be arranged on the shelf?

b. How many ways can the books be arranged so that Volume 5 will be the rightmost book?

c. Use the answers from 9a and b to find the probability that Volume 5 will be the rightmost book if the books are arranged at random.

d. Explain how to compute the probability in 9c using another method.

e. If the books are arranged randomly, what is the probability that the last book on the right is an even-numbered volume? Explain how you determined this probability.

f. How many ways can the books be arranged so that they are in the correct order, with volume numbers increasing from left to right?

g. How many ways can the books be arranged so that they are out of order?

h. What is the probability that the books happen to be in the correct order?

10. Suppose a computer is programmed to list all permutations of N items. Use the values given in the table below to figure out how long it will take for the computer to list all of the permutations for the remaining values of N listed. Use an appropriate time unit for each answer (minutes, hours, days, or years).

N	Number of permutations of N items	Time
5	120	0.00012 s
10	3,628,800	3.6288 s
12		
13		
15		
20		

Technology CONNECTION

Computers are programmed to play games such as chess by considering all possible moves. By working with a tree diagram of possible moves, a computer can choose from millions of board positions in a matter of seconds. A person, on the other hand, usually plays the game by remembering and visualizing previous winning moves. It usually takes many minutes for a human to decide which move to make.

Photographed here in 1997, champion chess player Garry Kasparov plays against IBM's chess-playing computer, Deep Blue. Kasparov lost the match.

11. You have purchased four tickets to a charity raffle. Only 50 tickets were sold. Three tickets will be drawn, and first, second, and third prizes will be awarded.

a. What is the probability that you win the first prize (and no other prize)?

b. What is the probability that you win both the first and second prizes, but not the third prize?

c. What is the probability that you win the second or third prize?

d. If the first, second, and third prizes are gift certificates for $25, $10, and $5, respectively, what is the expected value of your winnings?

12. **APPLICATION** Biologists use Punnett squares to represent the ways that genes can be passed from parent to offspring. In the Punnett squares at right, B stands for brown eyes, a dominant trait, whereas b stands for blue eyes, a recessive trait. E represents brown hair, the dominant trait, whereas e represents the recessive trait, blonde hair. If a dominant gene is present, a recessive trait will be hidden. The first row and column represent the parents' genes. The rest of the squares represent the possible pairs of genes the parents could pass on to their offspring.

a. In Table 1, both parents are brown-eyed, but one parent has a Bb gene combination, whereas the other has a BB gene combination. What is the probability of a blue-eyed (bb) offspring?

b. In Table 2, both parents are brown-eyed, each having the Bb gene combination. What is the probability of a blue-eyed (bb) offspring?

c. In Table 3, one parent is blue-eyed (bb), whereas the other is brown-eyed (Bb). What is the probability of a blue-eyed (bb) offspring?

d. According to Table 4, is it possible for two blue-eyed parents to have a child with brown eyes? Explain.

e. In Table 5, one parent is brown-eyed and brown-haired (BbEE). The other parent is brown-eyed and blonde-haired (Bbee). What is the probability of a blue-eyed, brown-haired child?

Table 1

	B	b
B	BB	Bb
B	BB	Bb

Table 2

	B	b
B	BB	Bb
b	Bb	bb

Table 3

	b	b
B	Bb	Bb
b	bb	bb

Table 4

	b	b
b	bb	bb
b	bb	bb

Table 5

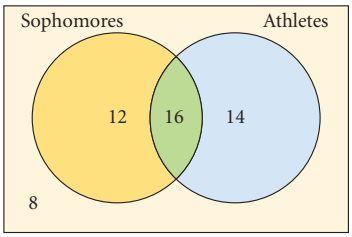
	BE	BE	bE	bE
Be	BBEe	BBEe	BbEe	BbEe
Be	BBEe	BBEe	BbEe	BbEe
be	BbEe	BbEe	bbEe	bbEe
be	BbEe	BbEe	bbEe	bbEe

Science CONNECTION

Genetics helps predict the likelihood of inheriting particular traits. Genetics can help animal breeders develop breeds with more desirable qualities than existing breeds have. And new strains of disease-resistant crops can be developed by crossbreeding existing plants. Genetic counselors can test the genes of human parents to see whether they are carriers of genetic diseases, such as sickle-cell anemia, Tay-Sachs disease, or hemophilia, which are passed on from generation to generation. To learn more about the science of genetics, visit the weblinks at www.keymath.com/DAA .

This photo shows an image of a karyotype, all 23 pairs of chromosomes found in every human cell. The genes in these chromosomes determine individual characteristics.

Review



13. In a gym class of 50 students, 28 are sophomores and 30 are athletes.

a. What is the probability that a randomly selected member of the class is an athlete?

b. If you already know a randomly selected member is an athlete, what is the probability that this student is a sophomore?

14. A fair six-sided die is rolled. If the number showing is even, you lose a point for each dot showing. If the number showing is odd, you win a point for each dot showing.

a. Find $E(x)$ for one roll if x represents the number of points you win.

b. Find the expected winnings for ten rolls.

15. Assume that boys and girls are equally likely to be born.

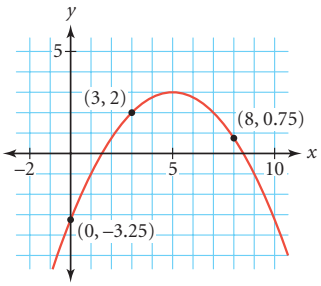
a. If there are three consecutive births, what is the probability of two girls and one boy being born, in that order?

b. What is the probability that two girls and one boy are born in any order?

c. Given that the first two babies born are girls, what is the probability that the third will be a boy?

16. Write the equation of the parabola at right in

a. Polynomial form. **b.** Vertex form. **c.** Factored form.



17. As an art project, Jesse is planning to make a set of nesting boxes (boxes that fit inside each other). Each box requires five squares that will fold into an open box with no lid. The largest box will measure 4 in. on each side. The side length of each successive box is 95% of the previous box. Jesse wants the smallest box to be no less than 0.5 in. on a side.

a. How many boxes can Jesse make in one set?

b. What is the total amount of paper needed for one set of open boxes?

IMPROVING YOUR VISUAL THINKING SKILLS

A Perfect Arrangement

Craig has a summer landscape business planting trees and shrubs. His neighbor, Mr. Malone, contracted with him to plant 5 rows of fruit trees with 4 trees in each row. He said that he had all the trees ready for him to plant. When Craig arrived to do the planting, he found 10 trees. He was about to point out the mistake, but just completed the job instead. What did he do?

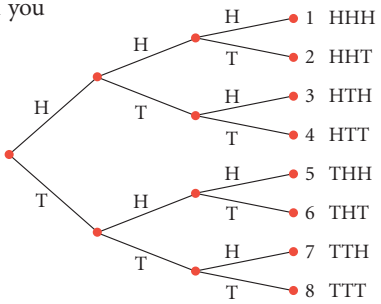
LESSON 12.6

Combinations and Probability

Math is like love—a simple idea but it can get complicated.

R. DRABEK

If three coins are flipped, a tree diagram and the counting principle both indicate that there are $\underline{2} \cdot \underline{2} \cdot \underline{2}$, or 8, equally likely outcomes: 2 choices, then 2 choices, then 2 choices. But if you are not concerned about the order in which the heads and tails occur, then you can describe paths 2, 3, and 5 as "2 heads and 1 tail," and paths 4, 6, and 7 as "1 head and 2 tails." So, if you're not concerned about order, there are only 4 outcomes, which are not equally likely:



3 heads (one path)
2 heads and 1 tail (three paths)
1 head and 2 tails (three paths)
3 tails (one path)

In this lesson you will learn about counting outcomes and calculating theoretical probabilities when order doesn't matter. There are generally fewer possibilities if order doesn't matter than there are if order is important.

EXAMPLE A

At the first meeting of the International Club, the members get acquainted by introducing themselves and shaking hands. Each member shakes hands with every other member exactly once. How many handshakes are there in each of the situations listed below?

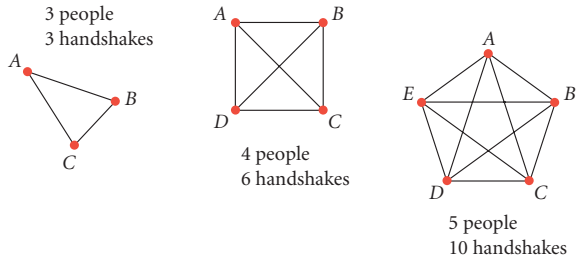
a. Three people meet.

b. Four people meet.

c. Five people meet.

d. Fifteen people meet.

▶ Solution



The points (vertices) pictured in the diagram above can represent the three, four, or five people in a room, and the lines (edges) can represent the handshakes. The diagrams show that there are 3 handshakes between 3 people, 6 handshakes between 4 people, and 10 handshakes between 5 people. You can find the number of handshakes by counting edges, but as you add more people to the group, it will become more difficult to draw and count. So, look for patterns to determine the number of handshakes between 15 people.

The four edges at each vertex in the five-person handshake diagram might suggest that there are $5 \cdot 4$ edges. However, if you use this method to count edges, then an edge like $\overline{DB}$ will be counted at vertex D and again at vertex B. Because $\overline{DB}$ is the same as $\overline{BD}$, you are counting twice as many edges as the actual total. The number of possibilities in which order doesn't matter is only half the number in which order does matter. Therefore, if 15 people are in the room, there are $\frac{15 \cdot 14}{2}$, or 105 handshakes.

You can think of each handshake as a pairing of two of the people in the room, or two of the vertices. When you count collections of people or objects, without regard to order, you are counting **combinations.**

The number of combinations of 5 people taken 2 at a time is symbolized by ${}_5C_2$. (The notation ${}_5C_2$ can be read as "five choose two.") Although there are ${}_5P_2$, or 20, permutations of 5 vertices taken 2 at a time, you have only half as many combinations:

$${}_5C_2 = \frac{{}_5P_2}{2} = \frac{20}{2} = 10$$

You can calculate the number of handshakes in Example A in the same way:

$${}_{15}C_2 = \frac{{}_5P_2}{2} = \frac{15 \cdot 14}{2} = 15 \cdot 7 = 105$$

There are 105 handshakes.

EXAMPLE B

Anna, Ben, Chang, and Dena are members of the International Club, and they have volunteered to be on a committee that will arrange a reception for exchange students. Usually there are only three students on the committee. How many different three-member committees could be formed with these four students?

▶ Solution

Note that order isn't important in these committees. ABD and BDA are the same committee and shouldn't be counted more than once. The number of different committee combinations will be fewer than the number of permutations, ${}_4P_3$, or 24, listed on the next page.

ABC	**ABD**	**ACD**	**BCD**
ACB	ADB	ADC	BDC
BAC	BAD	CAD	CBD
BCA	BDA	CDA	CDB
CAB	DAB	DAC	DBC
CBA	DBA	DCA	DCB

The four committees listed in the top row can represent all of the 3!, or 6, permutations listed in each column. Therefore, the number of combinations, ${}_4C_3$, is one-sixth the number of permutations. That is, ${}_4C_3 = \frac{{}_4P_3}{3!}$. You can evaluate ${}_4C_3$ using the factorial definition of ${}_nP_r$:

$$ {}_4C_3 = \frac{{}_4P_3}{3!} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!\,1!} = \frac{4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot 1} = 4 $$

Combinations

A **combination** is a grouping of some or all of the objects from a set without regard to order.

The number of combinations of n objects chosen r at a time ($r \leq n$) is

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!\,(n-r)!} $$

Rather than simply memorizing the formula given above, try to understand how numbers of combinations relate to numbers of permutations and to a tree diagram. You may want to draw a representation of each problem you investigate.

You can count combinations to calculate theoretical probabilities.

EXAMPLE C

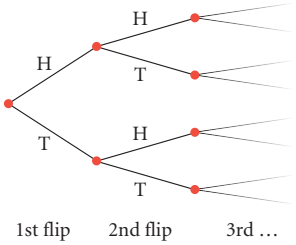
Suppose a coin is flipped 10 times.

a. What is the probability that it will land heads exactly five times?

b. What is the probability that it will land heads exactly five times, including on the third flip?

▶ Solution

The tree diagram for this problem has ten stages (one for each flip) and splits into two possibilities (heads or tails) at each point on the path. It's not necessary to draw the entire tree diagram. By the counting principle there are 2^{10} or 1024 possibilities.



a. To find the numerator of the probability ratio, you must determine how many of the 1024 separate paths contain 5 heads. Because order is not important, you can find the number of paths that fit this description by counting combinations. There are ${}_{10}C_5 = \frac{10!}{5!\,5!} = 252$ ways of choosing 5 of the 10 flips to contain H's. [▶ See **Calculator Note 12E** to learn how to find numbers of combinations on a calculator. ◀] That is, 252 of the 1024 paths will contain exactly 5 heads.

Therefore, the probability that you will get exactly 5 heads and 5 tails is $\frac{252}{1024}$, or approximately .246.

b. If there's a head on the third flip, then the other four heads must be chosen from the other nine flips. There are ${}_9C_4 = \frac{9!}{5!\,4!}$ or 126 ways of making that choice. Therefore, the probability of this event is $\frac{126}{1024}$, or approximately .123.

In the investigation you'll count combinations to discover mathematical reasons why playing the lottery is a losing proposition.

Investigation
Winning the Lottery

You will do Step 1 of this investigation with the whole class. Then, in Steps 2–8, you will work with your group to analyze the results.

Consider a state lottery called Lotto 47. Twice a week, players select six different numbers between 1 and 47. The state lottery commission also selects six numbers between 1 and 47. Selection order doesn't matter, but a player needs to match all six numbers to win Lotto 47.

Step 1 Follow these directions with your class to simulate playing Lotto 47.

a. For five minutes, write down as many sets of six different numbers as you can. Write only integers between 1 and 47.

b. After five minutes of writing, everyone stands up.

c. Your teacher will generate a random integer, 1–47. Cross out all of your sets of six numbers that do not contain the given number. If you cross out all of your sets, sit down.

d. Your teacher will generate a second number, 1–47. (If it's the same number as before, it will be skipped.) Again, cross out all of your sets that do not contain this number. If you cross out all of your sets, sit down.

e. Your teacher will continue generating different random numbers until no one is still standing, or six numbers have been generated.

Work together with your group to answer the questions in Steps 2–8.

Step 2 What is the probability that any one set of six numbers wins?

Step 3 At $1 for each set of six numbers, how much did each of your group members invest during the first five minutes? What was the total group investment?

Step 4 Estimate the total amount invested by the entire class during the first five minutes. Explain how you determined this estimate.

Step 5	Estimate the probability that someone in your class wins. Explain how you determined this estimate.
Step 6	Estimate the probability that someone in your school would win if everyone in the school participated in this activity. Explain how you determined this estimate.
Step 7	If each possible set of six numbers were written on a 1-inch chip, and if all the chips were laid end to end, how long would the line of chips be? Convert your answer to an appropriate unit.
Step 8	Write a paragraph comparing Lotto 47 with some other event whose probability is approximately the same.

EXERCISES

Practice Your Skills

1. Evaluate each expression without using your calculator.

a. $\frac{10!}{3!\,7!}$ **b.** $\frac{7!}{4!\,3!}$ **c.** $\frac{15!}{13!\,2!}$ **d.** $\frac{7!}{7!\,0!}$

2. Evaluate each expression.

a. ${}_{10}C_7$ **b.** ${}_7C_3$ **c.** ${}_{15}C_2$ **d.** ${}_7C_0$

3. Consider each expression in the form ${}_nP_r$ and ${}_nC_r$.

a. What is the relationship between ${}_7P_2$ and ${}_7C_2$?

b. What is the relationship between ${}_7P_3$ and ${}_7C_3$?

c. What is the relationship between ${}_7P_4$ and ${}_7C_4$?

d. What is the relationship between ${}_7P_7$ and ${}_7C_7$?

e. Describe how you can find ${}_nC_r$ if you know ${}_nP_r$.

Keymath.com More Practice Your Skills

4. Which is larger, ${}_{18}C_2$ or ${}_{18}C_{16}$? Explain.

Reason and Apply

5. For what value(s) of n and r does ${}_nC_r$ equal 35?

6. Find a number r, $r \neq 4$, such that ${}_{10}C_r = {}_{10}C_4$. Explain why your answer makes sense.

7. Suppose you need to answer any four of seven essay questions on a history test and you can answer them in any order.

a. How many different question combinations are possible?

b. What is the probability that you include Essay Question 5 if you randomly select your combination?

8. Does a "combination lock" really use combinations of numbers? Should it be called a "permutation lock?" Explain.

9. Find the following sums.
   a. ${}_2C_0 + {}_2C_1 + {}_2C_2$
   b. ${}_3C_0 + {}_3C_1 + {}_3C_2 + {}_3C_3$
   c. ${}_4C_0 + {}_4C_1 + {}_4C_2 + {}_4C_3 + {}_4C_4$
   d. Make a conjecture about the sum of ${}_nC_0 + {}_nC_1 + \cdots + {}_nC_n$. Test it by finding the sum for all possible combinations of five things.

10. Consider the Lotto 47 game, which you simulated in the Investigation Winning the Lottery.
   a. If it takes someone 10 seconds to fill out a Lotto 47 ticket, how long would it take him or her to fill out all possible tickets?
   b. If someone fills out 1000 tickets, what is his or her probability of winning Lotto 47?

11. Draw a circle and space points equally on its circumference. Draw chords to connect all pairs of points. How many chords are there if you place
   a. 4 points?
   b. 5 points?
   c. 9 points?
   d. n points?

12. In most state and local courts, 12 jurors and 2 alternates are chosen from a pool of 30 prospective jurors. The order of the alternates is specified. If a juror is unable to serve, then the first alternate will replace that juror. The second alternate will be called on if another juror is dismissed.
   a. In how many ways can 12 jurors and first and second alternates be chosen from 30 people?
   b. In federal court cases, 12 jurors and 4 alternates are usually selected from a pool of 64 prospective jurors. In how many ways can this be done?

Social Science CONNECTION

The U.S. Constitution guarantees a person accused of a crime the right to a speedy and public trial by an impartial jury. In the jury selection process, both the defense and the prosecution can "challenge" (request that the judge dismiss) potential jurors. Each side tries to seat a jury they feel will benefit their case. Often, jury consultants are brought in to evaluate prospective jurors and to investigate community and individual attitudes and biases.

A jury box

Review

13. Expand each expression.

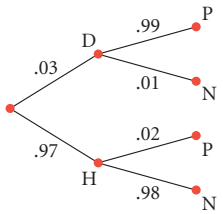
a. $(x + y)^2$ **b.** $(x + y)^3$ **c.** $(x + y)^4$

14. How many speeds, or combinations of gears, does a bicycle have if it has three gears in front and five gears in the rear?

15. Use the tree diagram at right. Find each probability and explain its meaning.

a. $P(\text{H and P})$ **b.** $P(\text{P} \mid \text{H})$

c. $P(\text{P})$ **d.** $P(\text{H} \mid \text{P})$



D = disease
H = healthy
P = tests positive
N = tests negative

16. If the equation $x^2 + y^2 + 6x - 11y + C = 0$ describes a circle, give the range of possible values for C.

17. While on a bird-watching field trip, Angelo sees a Boreal owl at the top of a tall tree. Lying on his stomach in the grass, he measures the angle of elevation of his line of sight to the owl to be 32°. He then crawls 8.6 m closer to the tree. The angle of his line of sight is now 42°. He doesn't move any closer for fear of disturbing the owl. How close did he get to the tree? How close did he get to the owl?

18. Suppose the value of a certain building depreciates at a rate of 6% per year. When new, the building was worth $36,500.

a. How much is the building worth after 5 years and 3 months?

b. To the nearest month, when will the building be worth less than $10,000?

John James Audubon (1785–1851) engraved this image of a Tengmalm's night owl, known as a Boreal owl in North America.

IMPROVING YOUR REASONING SKILLS

A Change of Plans

Angelo takes the train into the city for his weekly guitar lesson. After his lesson his mother picks him up at the train station knowing he will be on the 2:00 P.M. train. She always gets to the station just as the train arrives. This week when Angelo arrived for his lesson, he found out that his teacher was ill and had to cancel the lesson. Angelo went back to the train station and just managed to get the 1:00 P.M. train. He didn't have time to call his mother about the change. When he arrived at his station, he decided to start walking home until he met his mother en route to the station. When he saw her, he flagged his mother down, hopped into the car, and they arrived home 20 min ahead of their usual time. How long did Angelo walk before he met his mother?

LESSON

12.7

The Binomial Theorem and Pascal's Triangle

Mathematics is the science of patterns.

LYNN ARTHUR STEEN

Probability is an area of mathematics that is rich with patterns. Many random processes, such as flipping coins, involve patterns in which there are two possible outcomes. In this lesson you will learn about using the binomial theorem and Pascal's triangle to find probabilities in those cases.

Shown at left is **Pascal's triangle.** It contains many different patterns that have been studied for centuries. The triangle begins with a 1, then there are two 1's below it. Each successive row is filled in with numbers formed by adding the two numbers above it. For example, each of the 4's is the sum of a 1 and a 3 in the previous row. Every row begins and ends with a 1.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

In Lesson 12.6, you studied numbers of combinations. These numbers occur in the rows of Pascal's triangle. For example, the numbers 1, 5, 10, 10, 5, and 1 in the sixth row are the values of ${}_5C_r$:

$${}_5C_0 = 1 \quad {}_5C_1 = 5 \quad {}_5C_2 = 10 \quad {}_5C_3 = 10 \quad {}_5C_4 = 5 \quad {}_5C_5 = 1$$

Is this the case in all rows? If so, why? In the investigation you'll explore these questions.

Investigation

Pascal's Triangle and Combination Numbers

A group of five students regularly eats lunch together, but each day only three of them can show up.

Step 1 How many groups of three students could there be? Express your answer in the form ${}_nC_r$ and as a numeral.

Step 2 If Leora is definitely at the table, how many other students are at the table? How many students are there to choose from? Find the number of combinations of students possible in this instance. Express your answer in the form ${}_nC_r$ and as a numeral.

Step 3 How many combinations are there that don't include Leora? Consider how many students there are to select from and how many are to be chosen. Express your answer in the form ${}_nC_r$ and as a numeral.

Step 4 Repeat Steps 1–3 for groups of four of the five students.

Step 5 | What patterns do you notice in your answers to Steps 1–3 for groups of three students and four students? Write a general rule that expresses ${}_nC_r$ as a sum of other combination numbers.

Step 6 | How does this rule relate to Pascal's triangle?

Pascal's triangle can provide a shortcut for expanding binomials. An **expansion** of a binomial raised to a power is the power expressed as a single polynomial. For example, the expansion of $(x + y)^3$ is $1x^3 + 3x^2y + 3xy^2 + 1y^3$. Note that the coefficients of this expansion are the numbers in the fourth row of Pascal's triangle.

Why are the numbers in Pascal's triangle equal to the coefficients of a binomial expansion? Remember that the numbers in Pascal's triangle are evaluations of ${}_nC_r$. So, the question could also be asked, why are the coefficients of a binomial expansion equal to values of ${}_nC_r$?

EXAMPLE A | Expand $(\mathrm{H} + \mathrm{T})^3$. Relate the coefficients in the expansion to combinations.

▶ Solution

You can write $(\mathrm{H} + \mathrm{T})^3$ as $(\mathrm{H} + \mathrm{T})(\mathrm{H} + \mathrm{T})(\mathrm{H} + \mathrm{T})$. To expand this product, multiply in steps. First you multiply the first binomial, $(\mathrm{H} + \mathrm{T})$, by the second binomial, $(\mathrm{H} + \mathrm{T})$:

$$(\mathrm{H} + \mathrm{T})(\mathrm{H} + \mathrm{T}) = (\mathrm{HH} + \mathrm{HT} + \mathrm{TH} + \mathrm{TT})$$

Then you multiply each term of that result with the H and the T in the third binomial:

$$(\mathrm{HH} + \mathrm{HT} + \mathrm{TH} + \mathrm{TT})(\mathrm{H} + \mathrm{T}) =$$
$$\mathrm{HHH} + \mathrm{HHT} + \mathrm{HTH} + \mathrm{HTT} + \mathrm{THH} + \mathrm{THT} + \mathrm{TTH} + \mathrm{TTT}$$

Notice that if H represents flipping a head, and T represents flipping a tail, this result shows you all the possible outcomes of flipping a coin three times. If the order does not matter, you can also write this expression as

$$1\mathrm{H}^3 + 3\mathrm{H}^2\mathrm{T} + 3\mathrm{HT}^2 + 1\mathrm{T}^3$$

This expression shows that there is one way of getting three heads, there are three ways of getting two heads and a tail, and so on. You can also express these numbers of combinations as ${}_3C_3$, ${}_3C_2$, and so on.

Here are binomial expansions of the first few powers of $(\mathrm{H} + \mathrm{T})$. Notice that the coefficients are the same as the rows in Pascal's triangle. Think about what these expansions tell you about the results of flipping a coin 0, 1, 2, and 3 times.

$(\mathrm{H} + \mathrm{T})^0 =$	$\mathbf{1}$	1st row
$(\mathrm{H} + \mathrm{T})^1 =$	$\mathbf{1}\mathrm{H} + \mathbf{1}\mathrm{T}$	2nd row
$(\mathrm{H} + \mathrm{T})^2 =$	$\mathbf{1}\mathrm{H}^2 + \mathbf{2}\mathrm{HT} + \mathbf{1}\mathrm{T}^2$	3rd row
$(\mathrm{H} + \mathrm{T})^3 =$	$\mathbf{1}\mathrm{H}^3 + \mathbf{3}\mathrm{H}^2\mathrm{T} + \mathbf{3}\mathrm{HT}^2 + \mathbf{1}\mathrm{T}^3$	4th row

Notice the pattern in the coefficients. You can use Pascal's triangle, or the values of ${}_nC_r$, to expand binomials without multiplying all the terms together.

The Binomial Theorem

If a binomial $(p + q)$ is raised to a whole number power, n, the coefficients of the expansion are the combination numbers ${}_nC_n$ to ${}_nC_0$, as shown:

$$(p + q)^n = {}_nC_n p^n q^0 + {}_nC_{(n-1)} p^{n-1} q^1 + {}_nC_{(n-2)} p^{n-2} q^2 + \cdots + {}_nC_0 p^0 q^n$$

or

$$(p + q)^n = \sum_{j=0}^{n} {}_nC_{(n-j)} \cdot p^{n-j} q^j$$

You can use binomial expansions to represent results of random processes with two possible outcomes. In Example A, you saw that if a coin is flipped three times, these outcomes are possible: 1 outcome of 3 heads, 3 outcomes of 2 heads and 1 tail, 3 outcomes of 1 head and 2 tails, and 1 outcome of 3 tails.

When you flip a fair coin, the outcomes H and T are equally likely. Example B shows how you can use a binomial expansion to find probabilities of outcomes that are not equally likely.

EXAMPLE B

Suppose that a hatching yellow-bellied sapsucker has a .58 probability of surviving to adulthood. If a nest has 6 eggs, what are the probabilities that 0, 1, 2, 3, 4, 5, and 6 birds will survive?

A yellow-bellied sapsucker

▶ Solution

Represent the event of survival, or success, with S and the event of nonsurvival with N. Then let s represent $P(\text{S})$, the probability of a bird's survival, and let n represent $P(\text{N})$. The outcome that 4 of the 6 birds survive and the other 2 do not survive is given by S^4N^2. The number of combinations in which 4 birds survive is given by ${}_6C_4$. You can see the expression ${}_6C_4\text{S}^4\text{N}^2$ in the expansion of $(\text{S} + \text{N})^6$.

$$(\text{S} + \text{N})^6 = {}_6C_6\text{S}^6\text{N}^0 + {}_6C_5\text{S}^5\text{N}^1 + {}_6C_4\text{S}^4\text{N}^2 + {}_6C_3\text{S}^3\text{N}^3 + {}_6C_2\text{S}^2\text{N}^4 + {}_6C_1\text{S}^1\text{N}^5 + {}_6C_0\text{S}^0\text{N}^6$$

In fact, each term of the binomial expansion gives a number of ways that a certain number of birds will survive and the others will not. You can write a similar expansion using s and n to find the probability of each of those combinations. You can then substitute .58 for s and .42 for n to find the probability of each outcome.

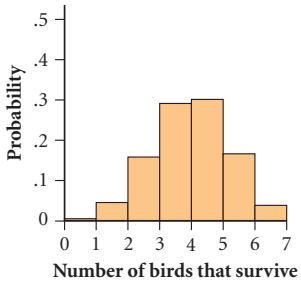
$$\begin{aligned}(s + n)^6 &= {}_6C_6 s^6 n^0 + {}_6C_5 s^5 n^1 + {}_6C_4 s^4 n^2 + {}_6C_3 s^3 n^3 + {}_6C_2 s^2 n^4 + {}_6C_1 s^1 n^5 + {}_6C_0 s^0 n^6 \\ &= 1(.58)^6(.42)^0 + 6(.58)^5(.42)^1 + 15(.58)^4(.42)^2 + 20(.58)^3(.42)^3 + 15(.58)^2(.42)^4 + 6(.58)^1(.42)^5 + 1(.58)^0(.42)^6 \\ &\approx .038 + .165 + .299 + .289 + .157 + .045 + .005\end{aligned}$$

You can put these numbers into a table to organize the information.

Number of birds that survive x	6	5	4	3	2	1	0
Probability $P(x)$	.038	.165	.299	.289	.157	.045	.005

So, it is most likely that 4 birds survive and very unlikely that 0 will survive.

You could show these probabilities with a histogram, as shown at right. This display allows you to compare the probabilities of different outcomes quickly. Notice the general shape of the histogram. You'll learn more about probability diagrams like these in Chapter 13.



In a situation like Example B, sometimes you want to know the probability that at most 4 birds survive or that at least 4 birds survive. How could you calculate the values in this expanded table? You will calculate the missing table values in Exercise 3.

	6 birds	**5 birds**	**4 birds**	**3 birds**	**2 birds**	**1 bird**	**0 birds**
Exactly	.038	.165		.289	.157	.045	.005
At most	1	.962			.207	.050	.005
At least	.038				.949	.995	1

You can think of the 6 birds as a **sample** taken from a **population** of many birds. You know the probability of success in the population, and you want to calculate various probabilities of success in the sample. You saw in Example B that a survival rate of 4 birds was most likely.

What if the situation is reversed? Suppose you don't know what the probability of success in the population is, but you do know that on average 4 out of 6 birds survive to adulthood. This ratio would indicate a population success probability of $\frac{4}{6}$, or .67; but you saw in the example that this ratio is also highly likely with a population success probability of .58. What range of population success probabilities would have the highest probability of 4 survivors out of 6?

This kind of problem often presents itself when voters are polled. A pollster questions a sample of voters, not the entire population. Statisticians then have to interpret the data from that sample to make predictions about the population.

EXAMPLE C

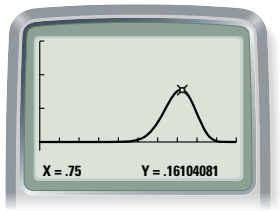
A random sample of 32 voters is taken from a large population. In the sample, 24 voters favor passing a proposal. What is the probability that a randomly selected member of the whole population favors the proposal? What can you predict about the whole population?

▶ Solution

The ratio of supporters to voters in the sample is $\frac{24}{32}$, or 0.75. This ratio of support in the sample is highly likely if the probability that a randomly chosen member of the population supports the proposal is also .75. But it could also occur for other probabilities of success in the population.

You can represent the unknown probability of support in the population with p and the probability of nonsupport with q. Because p and q are complements, the value of q is $1 - p$. The probability that exactly n people in a sample of 32 will be supporters is ${}_{32}C_n p^n q^{32-n}$.

If $n = 24$, as in the sample taken, then the most likely value of p is .75. In fact, if you graph $y = {}_{32}C_{24}p^{24}(1 - p)^8$, the maximum value of the function is at $p = .75$.



But there may be other values of p for which 24 supporters are likely. What other values of p give a higher probability of 24 supporters than of 23 or 25 supporters? That is, what values of p will satisfy these two inequalities?

$${}_{32}C_{24}p^{24}q^8 > {}_{32}C_{23}p^{23}q^9 \quad \text{and} \quad {}_{32}C_{24}p^{24}q^8 > {}_{32}C_{25}p^{25}q^7$$

You can solve the first inequality using algebra.

${}_{32}C_{24}p^{24}q^8 > {}_{32}C_{23}p^{23}q^9$	Original inequality.
${}_{32}C_{24}p > {}_{32}C_{23}q$	Divide both sides by p^{23} and by q^8.
$\frac{32!}{24!\,8!}p > \frac{32!}{23!\,9!}q$	Substitute numbers of combinations for ${}_{32}C_{24}$ and ${}_{32}C_{23}$.
$\frac{p}{24!\,8!} > \frac{q}{23!\,9!}$	Divide both sides by 32!.
$9p > 24q$	Multiply both sides by 24! and by 9!.
$9p > 24(1 - p)$	Replace q with $1 - p$.
$33p > 24$	Add $24p$ to both sides.
$p > \frac{24}{33}$	Divide both sides by 33.

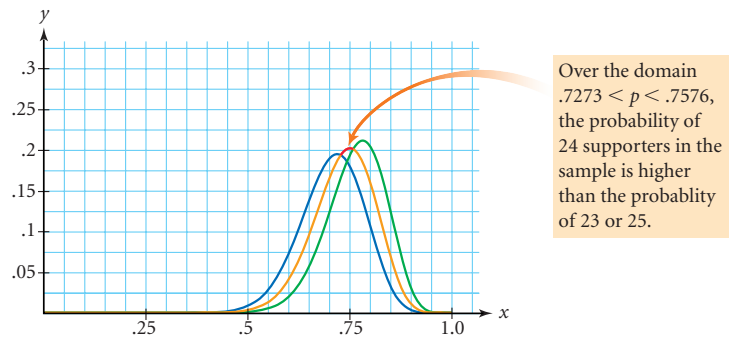
Solving the other inequality in the same way gives

$$p < \frac{25}{33}$$

You can then say that

$$\frac{24}{33} < p < \frac{25}{33}, \text{ or } .7273 < p < .7576$$

Compare the graph on page 714 with the graphs of $y = {}_{32}C_{23}p^{23}(1 - p)^9$ and $y = {}_{32}C_{25}p^{25}(1 - p)^7$, shown below. You find that the inequality corresponds to the portion of the graph where the value of the original function is higher than either of these two functions.



So, a random member of the population is most likely to have a probability between about 73% and 75% of favoring the proposal. The proposal is likely to pass.

In Example C, you found an interval for a population probability. In Chapter 13, you will learn about another type of interval based on probability.

EXERCISES

You will need

Statistics software for Exercise 18

Practice Your Skills

1. Given the expression $(x + y)^{47}$, find the terms below.

a. 1st term **b.** 11th term **c.** 41st term **d.** 47th term

2. If the probability of success for each trial is .25 and all trials are independent, then

a. What is the probability of failure for a single trial?

b. What is the probability of two successes in two trials?

c. What is the probability of n successes in n trials?

d. What is the probability that there will be some combination of two successes and three failures in five trials?

3. Return to the expanded table after Example B on page 713.

a. Fill in the missing probability in the "exactly" row.

b. Find the two missing probabilities in the "at most" row. For example, to find the probability that *at most* 2 birds survive, you could find the sum of the probabilities of 0, 1, and 2 birds surviving.

c. Find the three missing probabilities in the "at least" row.

d. Why don't the "at most" and "at least" values for each number of birds sum to 1?

e. Make a statement about birds that incorporates the 20.3% entry in the "at least" row.

4. Suppose that the probability of success is .62. What is the probability that there are 35 successes in 50 trials?

5. Solve for p: ${}_{32}C_{24}p^{24}q^{8} > {}_{32}C_{25}p^{25}q^{7}$

Reason and Apply

6. Answer each probability question.
   a. List the equally likely outcomes if a coin is tossed twice.
   b. List the equally likely outcomes if two coins are tossed once.
   c. Draw a tree diagram that illustrates the answers to 6a and b.
   d. Describe the connection between the expressions ${}_2C_0 = 1$, ${}_2C_1 = 2$, and ${}_2C_2 = 1$, and the results of your answers to 6a–c.
   e. Give a real-world meaning to the equation
   $$(\mathrm{H} + \mathrm{T})^2 = 1\mathrm{H}^2 + 2\mathrm{HT} + 1\mathrm{T}^2$$

7. Expand each binomial expression.
   a. $(x + y)^4$ b. $(p + q)^5$ c. $(2x + 3)^3$ d. $(3x - 4)^4$

8. **APPLICATION** A survey of 50 people shows that only 10 support a new traffic circle.
   a. Which term of $(p + q)^{50}$ would correspond to the results of this sample?
   b. What inequality does p satisfy if the probability that 10 of the 50 are supporters is more than the probability that 11 are supporters?
   c. What inequality does p satisfy if the probability that 10 of the 50 are supporters is more than the probability that 9 are supporters?
   d. Solve these inequalities to find an interval for p.

9. Dr. Miller is using a method of treatment that is 97% effective.
   a. What is the probability that there will be no failure in 30 treatments?
   b. What is the probability that there will be fewer than 3 failures in 30 treatments?
   c. Let x represent the number of failures in 30 treatments. Write an equation that will provide a table of values representing the probability $P(x)$ for any value of x.
   d. Use the equation and table from 9c to find the probability that there will be fewer than 3 failures in 30 treatments.

10. A university medical research team has developed a new test that is 88% effective at detecting a disease in its early stages. What is the probability that there will be more than 20 incorrect readings in 100 applications of the test on subjects known to have the disease? [▶ See **Calculator Note 12F** to learn how to find the terms of a binomial expansion. ◀]

11. Suppose the probability is .12 that a randomly chosen penny was minted before 1975. What is the probability that you will find 25 or more such coins in
   a. A roll of 100 pennies?
   b. Two rolls of 100 pennies each?
   c. Three rolls of 100 pennies each?

12. Suppose that a blue-footed booby has a 47% chance of surviving from egg to adulthood. For a nest of four eggs

a. What is the probability that all four birds will hatch and survive to adulthood?

b. What is the probability that none of the four birds will hatch and survive to adulthood?

c. How many birds would you expect to survive?

A pair of blue-footed boobies

13. A fair coin is tossed five times and comes up heads four out of five times. In your opinion, is this event a rare occurrence? Defend your position.

14. Data collected over the last ten years show that, in a particular town, it will rain sometime during 30% of the days in the spring. How likely is it that there will be a week with

a. Exactly five rainy days?

b. Exactly six rainy days?

c. Exactly seven rainy days?

d. At least five rainy days?

15. Consider the function $y = f(x) = \left(1 + \frac{1}{x}\right)^x$. Note that the right side of the equation is a binomial raised to a power.

a. Fill in the table below using the Binomial Theorem or Pascal's triangle. Verify using your calculator.

x	1	2	3	4
y				

b. Using your calculator, find $f(10)$, $f(100)$, $f(1000)$, and $f(10000)$.

c. Describe what happens to the values of $f(x)$ as you use larger and larger values of x.

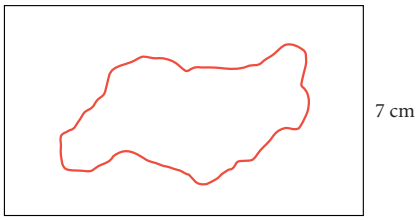
Mathematics CONNECTION

The real number e is a mathematical constant whose value is 2.718... . When a quantity changes at a rate proportional to the quantity present, the growth can be modeled using an exponential function with base e. This kind of change occurs when money is continuously compounded, when a capacitor discharges, and when a radioactive compound decays. In statistics, e is part of the equation that gives us the normal distribution curve, and in engineering, e is part of the equation of a *catenary,* a hanging cable. In the study of calculus, e is used most often in the study of exponents and logarithms.

16. Mrs. Gutierrez has 25 students and she sends 4 or 5 students at random to the board each day to solve a homework problem. You can calculate that there are ${}_{25}C_4$, or 12,650, ways that 4 students could be selected, and ${}_{25}C_5$, or 53,130, ways that 5 students could be selected. Suppose the class has grown to 26 students. Without using a calculator, determine how many ways 5 students can be selected now. Explain your solution method.

Review

17. Suppose that 350 points are randomly selected within the rectangle at right and 156 of them fall within the closed curve. What is an estimate of the area within the curve?



18. ***Technology*** Use statistics software to simulate flipping a coin. (See the Exploration The Law of Large Numbers on page 677 for help creating a simulation.)

a. Simulate 10 flips of a coin and make a bar graph of the results. How do your experimental probabilities compare to the theoretical probabilities of getting heads or tails?

b. Simulate 1000 flips of a coin by adding 990 more trials to your simulation. How do the experimental probabilities now compare to the theoretical probabilities?

c. Explain how you could modify your simulation to model an unfair coin that comes up heads more frequently than tails. Make the necessary modifications and test whether or not your simulation produces appropriate experimental results.

19. **APPLICATION** Kepler's third law of planetary motion states that the square of a planet's period is proportional to the cube of its average distance to the Sun. Planetary data are given in the table.

Planet	Average distance from Sun (miles)	Period of revolution (days)
Mercury	35,900,000	88
Venus	67,200,000	224.7
Earth	92,960,000	365.26
Mars	141,600,000	687
Jupiter	483,600,000	4,332.6
Saturn	886,700,000	10,759.2

a. Use logarithms and curve straightening to find an equation that fits data in the form (*distance, period*). Show all steps leading to your answer.

b. Use your equation to verify the data for Uranus, Neptune, or Pluto. How can you explain any errors?

Planet	Average distance from Sun (miles)	Period of revolution (days)
Uranus	1,783,000,000	30,685.4
Neptune	2,794,000,000	60,189
Pluto	3,666,100,000	90,465

c. Rewrite your equation from 19a in the form $p^2 = ka^3$.

Science CONNECTION

In 1618, after working for over 10 years with the data, German astronomer Johannes Kepler (1571–1630) suddenly realized that "the proportion between the periodic times of any two planets is precisely one and a half times the proportion of the mean distances." English physicist and mathematician Sir Isaac Newton (1642–1727) later restated Kepler's third law to include the masses of the planets. Newton was then able to calculate the masses of the planets relative to the mass of Earth.

CHAPTER 12 REVIEW

Keymath.com Links to Resources

In this chapter you were introduced to the concept of randomness and you learned how to generate random numbers on your calculator. Random numbers should all have an equal chance of occurring and should, in the long run, occur equally frequently. You can use random-number procedures to simulate situations and to determine the **experimental probability** of an **event.** You can determine **theoretical probability** by comparing the number of successful **outcomes** to the total number of possible outcomes. You represented situations involving probability with Venn diagrams and learned the meaning of events that are **dependent, independent,** and **mutually exclusive.** You used **tree diagrams** to help you count possibilities, and learned that sometimes probability situations can be represented geometrically. Using what you learned about theoretical probability, you were able to calculate **expected value,** by multiplying the value of each event by its probability and then summing all the products.

To help find theoretical probabilities, you were introduced to some formal counting techniques. The **counting principle** states that when there are n_1 ways to make the first choice, n_2 ways to make the second choice, n_3 ways to make the third choice, and so on, the product $n_1 \cdot n_2 \cdot n_3 \cdot \cdots$ represents the total number of different ways in which the entire sequence of choices can be made. These arrangements of choices, in which the order is important, are called **permutations.** The notation ${}_nP_r$ indicates the number of ways of choosing r things out of n possible choices. If the order is unimportant, then arrangements are called **combinations** and the notation ${}_nC_r$ represents the number of combinations of r things from a set of n choices. Your calculator can calculate permutation numbers and combination numbers, but before you use the calculator, be sure that you understand the situation and can visualize the possibilities. Combinations also appear in **Pascal's triangle** and as coefficients in **binomial expansions** that you can use to help calculate probabilities when there are two possible outcomes.

EXERCISES

1. Name two different ways to generate random numbers from 0 to 10.

2. Suppose you roll two octahedral (eight-sided) dice, numbered 1–8.

a. Draw a diagram that shows all possible outcomes of this experiment.

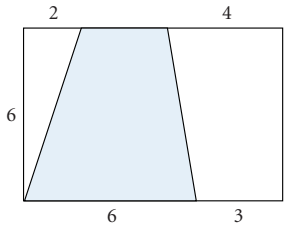
b. Indicate on your diagram all the possible outcomes for which the sum of the dice is less than 6.

c. What is the probability that the sum is less than 6?

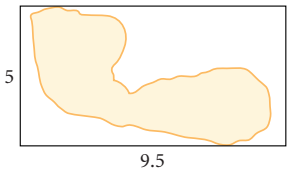
d. What is the probability that the sum is more than 6?

3. Answer each geometric probability problem.

a. What is the probability that a randomly plotted point will land in the shaded region pictured at right?



b. One thousand points are randomly plotted in the rectangular region shown below. Suppose that 374 of the points land in the shaded portion of the rectangle. What is an approximation of the area of the shaded portion?



4. A true-false test has five questions.

a. Draw a tree diagram representing all of the possible results. (Assume all five questions are answered.)

b. How many possible ways are there of getting three true and two false answers?

c. How could you use combinations or permutations to answer 4b?

d. Suppose you are sure that the answers to the first two questions on the test are true, and you write these answers down. Then you guess the answers to the remaining three questions at random. What is the probability that there will be three true and two false answers on the test?

5. The local outlet of Frankfurter Franchise sells three types of hot dogs: plain, with chili, and veggie. The owners know that 47% of their sales are chili dogs, 36% are plain, and the rest are veggie. They also offer three types of buns: plain, rye, and multigrain. Sixty-two percent of their sales are plain buns, 27% are multigrain, and the rest are rye. Assume that the choice of bun is independent of the choice of type of hot dog.

a. Make a tree diagram showing this information.

b. What is the probability that the next customer will order a chili dog on rye?

c. What is the probability that the next customer will *not* order a veggie dog on a plain bun?

d. What is the probability that the next customer will order either a plain hot dog on a plain bun or a chili dog on a multigrain bun?

6. All students in a school were surveyed regarding their preference for whipped cream or ice cream to be served with chocolate cake. The results, tabulated by grade level, are reported in the table.

	9th grade	10th grade	11th grade	12th grade	Total
Ice cream	18	37	85	114	
Whipped cream	5	18	37	58	
Total					

a. Copy and complete the table.

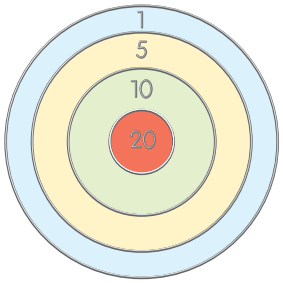
b. What is the probability that a randomly chosen 10th grader will prefer ice cream?

c. What is the probability that a randomly chosen 11th grader will prefer whipped cream?

d. What is the probability that someone who prefers ice cream is a 9th grader?

e. What is the probability that a randomly chosen student will prefer whipped cream?

7. Rita is practicing darts. On this particular dartboard, she can score 20 points for a bull's-eye and 10 points, 5 points, or 1 point for the other regions. Although Rita doesn't know exactly where her five darts will land, she has been a fairly consistent dart player over the years. She figures that she hits the bull's-eye 30% of the time, the 10-point circle 40% of the time, the 5-point circle 20% of the time, and the 1-point circle 5% of the time. What is the expected value of her score if she throws a dart ten times?



8. Misty polls residents of her neighborhood about the types of pets they have: cat, dog, other, or none. She determines these facts:

Ownership of cats and dogs is mutually exclusive.

32% of homes have dogs.

54% of homes have a dog or a cat.

16% of homes have only a cat.

42% of homes have no pets.

22% of homes have pets that are not cats or dogs.

Draw a Venn diagram of these data. Label each region with the probability of each outcome.

9. Elliott has time to take exactly 20 more pizza orders before closing. He has enough pepperoni for 16 more pizzas. On a typical night, 65% of orders are for pepperoni pizzas. What is the probability that Elliott will run short on pepperoni?

10. Find the term specified for each binomial expansion.

a. the first term of $\left(1 + \frac{x}{12}\right)^{99}$

b. the last term of $\left(1 + \frac{x}{12}\right)^{99}$

c. the tenth term of $(a + b)^{21}$

TAKE ANOTHER LOOK

1. Pascal's triangle is filled with patterns. Write the first ten rows of Pascal's triangle, and look for as many relationships as you can among the numbers. You may want to consider numerical patterns, sums of numbers in particular locations, and patterns of even or odd numbers, or numbers that share a factor.

2. In this chapter you have seen that a binomial expansion can help you find the probability of an outcome of a series of events when each event has only two possible results (such as success or failure). When there are three possible results instead of two, you can use trinomials in a similar way. Expand each trinomial given below. The first answer will have 6 terms, the second will have 10 terms, and the third will have 15 terms.

$(x + y + z)^2$ $\quad$ $(x + y + z)^3$ $\quad$ $(x + y + z)^4$

Write a formula for the expansion of $(x + y + z)^n$.

Assessing What You've Learned

GIVE A PRESENTATION By yourself or with a group, write and present a probability exercise. Detail the outcome you want to find the probability of, and tell how to solve the problem. If possible, explain how to find both experimental and theoretical probabilities of the outcome occurring. You may even want to do a simulation with the class. Or, present your work on a Project or Take Another Look from this chapter.

WRITE IN YOUR JOURNAL You have seen how to represent situations involving probabilities using tree diagrams and Venn diagrams. Describe situations in which each of these approaches would be appropriate, and explain how each method can help you solve problems. Specify the kinds of questions that can be answered with each type of diagram.

UPDATE YOUR PORTFOLIO Choose a couple of investigations or exercises from this chapter that you are particularly proud of. Write a paragraph about each piece of work. Describe the objective of the problem, how you demonstrated understanding in your solution, and anything you might have done differently.