CHAPTER 4

Functions, Relations, and Transformations

American artist Benjamin Edwards (b 1970) used a digital camera to collect images of commercial buildings for this painting, *Convergence.* He then projected all the images in succession on a 97-by-146-inch canvas, and filled in bits of each one. The result is that numerous buildings are transformed into one busy impression—much like the impression of seeing many things quickly out of the corner of your eye when driving through a city.

OBJECTIVES

In this chapter you will

- interpret graphs of functions and relations
- review function notation
- learn about the linear, quadratic, square root, absolute-value, and semicircle families of functions
- apply transformations—translations, reflections, stretches, and shrinks—to the graphs of functions and relations
- transform functions to model real-world data

LESSON 4.1

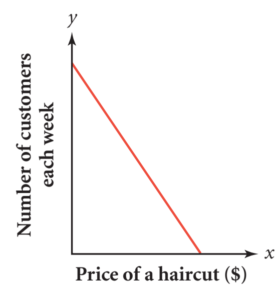
Interpreting Graphs

A picture can be worth a thousand words, if you can interpret the picture. In this lesson you will investigate the relationship between real-world situations and graphs that represent them.

Wigs (portfolio) (1994), by American artist Lorna Simpson (b 1960), uses photos of African-American hairstyles through the decades, with minimal text, to critique deeper issues of race, gender, and assimilation.

Lorna Simpson, *Wigs (portfolio),* 1994, waterless lithograph on felt, 72 x 162″ overall installed. Collection Walker Art Center, Minneapolis/T. B. Walker Acquisition Fund, 1995

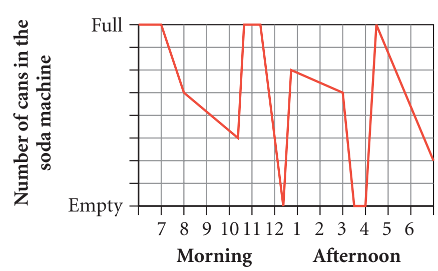
What is the real-world meaning of the graph at right, which shows the relationship between the number of customers getting haircuts each week and the price charged for each haircut?



The number of customers depends on the price of the haircut. So the price in dollars is the independent variable and the number of customers is the dependent variable. As the price increases, the number of haircuts decreases linearly. As you would expect, fewer people are willing to pay a high price; a lower price attracts more customers. The slope indicates the number of haircuts lost for each dollar increase. The x-intercept represents the haircut price that is too high for anyone. The y-intercept indicates the number of haircuts when they are free.

EXAMPLE

Students at Central High School are complaining that the soda pop machine is frequently empty. Several student council members decide to study this problem. They record the number of cans in the soda machine at various times during a typical school day and make a graph.



a. Based on the graph, at what times is soda consumed most rapidly?

b. When is the machine refilled? How can you tell?

c. When is the machine empty? How can you tell?

d. What do you think the student council will recommend to solve the problem?

▸ *Solution*

Each horizontal segment indicates a time interval when soda does not sell. Negative slopes represent when soda is consumed, and positive slopes show when the soda machine is refilled.

a. The most rapid consumption is pictured by the steep, negative slopes from 11:30 to 12:30, and from 3:00 to 3:30.

b. The machine is completely refilled overnight, again at 10:30 A.M., and again just after school lets out. The machine is also refilled at 12:30, but only to 75% capacity.

c. The machine is empty from 3:30 to 4:00 P.M., and briefly at about 12:30.

d. The student council might recommend refilling the machine once more at about 2:00 or 3:00 P.M. in order to solve the problem of it frequently being empty. Refilling the machine completely at 12:30 may also solve the problem.

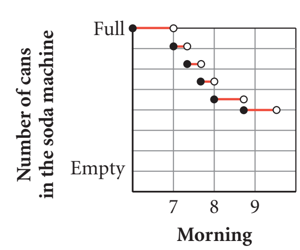
Health CONNECTION

Many school districts and several states have banned vending machines and the sale of soda pop and junk foods in their schools. Proponents say that schools have a responsibility to promote good health. The U.S. Department of Agriculture already bans the sale of foods with little nutritional value, such as soda, gum, and popsicles, in school cafeterias, but candy bars and potato chips don't fall under the ban because they contain some nutrients.

These recycled aluminum cans are waiting to be melted and made into new cans. Although 65% of the United States' aluminum is currently recycled, one million tons are still thrown away each year.

Although the student council members in the example are interested in solving a problem related to soda consumption, they could also use the graph to answer many other questions about Central High School: When do students arrive at school? What time do classes begin? When is lunch? When do classes let out for the day?

Both the graph of haircut customers and the graph in the example are shown as continuous graphs. In reality, the quantity of soda in the machine can take on only discrete values, because the number of cans must be a whole number. The graph might more accurately be drawn with a series of short horizontal segments, as shown at right. The price of a haircut and the number of haircuts can also take on only discrete values. This graph might be more accurately drawn with separate points. However, in both cases, a continuous "graph sketch" makes it easier to see the trends and patterns.



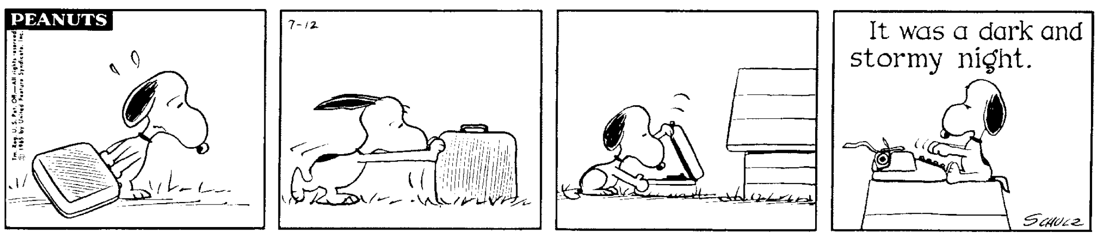
Investigation
Graph a Story

Every graph tells a story. Make a graph to go with the story in Part 1. Then invent your own story to go with the graph in Part 2.

Part 1

Sketch a graph that reflects all the information given in this story.

"It was a dark and stormy night. Before the torrents of rain came, the bucket was empty. The rain subsided at daybreak. The bucket remained untouched through the morning until Old Dog Trey arrived as thirsty as a dog. The sun shone brightly through the afternoon. Then Billy, the kid next door, arrived. He noticed two plugs in the side of the bucket. One of them was about a quarter of the way up, and the second one was near the bottom. As fast as you could blink an eye, he pulled out the plugs and ran away."



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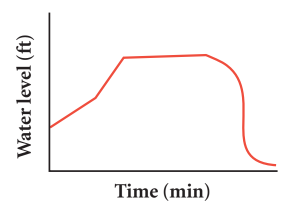
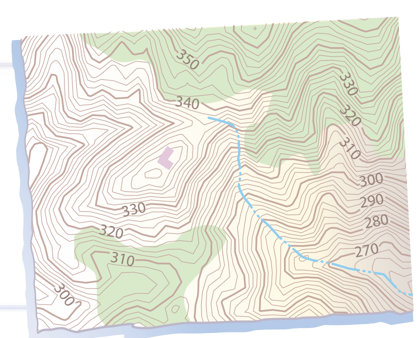
Part 2

This graph tells a story. It could be a story about a lake, a bathtub, or whatever you imagine. Spend some time with your group discussing the information contained in the graph. Write a story that conveys all of this information, including when and how the rates of change increase or decrease.



Science CONNECTION

Contour maps are a way to graphically represent altitude. Each line marks all of the points that are the same height in feet (or meters) above sea level. Using the distance between two contour lines, you can calculate the rate of change in altitude. These maps are used by hikers, forest fire fighters, and scientists.



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As you interpret data and graphs that show a relationship between two variables, you must always decide which is the independent variable and which is the dependent variable. You should also consider whether the variables are discrete or continuous.

EXERCISES

Practice Your Skills

1. Sketch a graph to match each description.

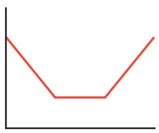
a. increasing throughout, first slowly and then at a faster rate

b. decreasing slowly, then more and more rapidly, then suddenly becoming constant

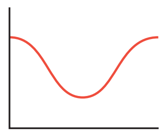
c. alternately increasing and decreasing without any sudden changes in rate

2. For each graph, write a description like those in Exercise 1.

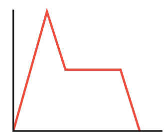
a.



b.

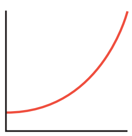


c.

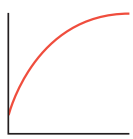


3. Match a description to each graph.

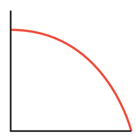
a.



b.



c.



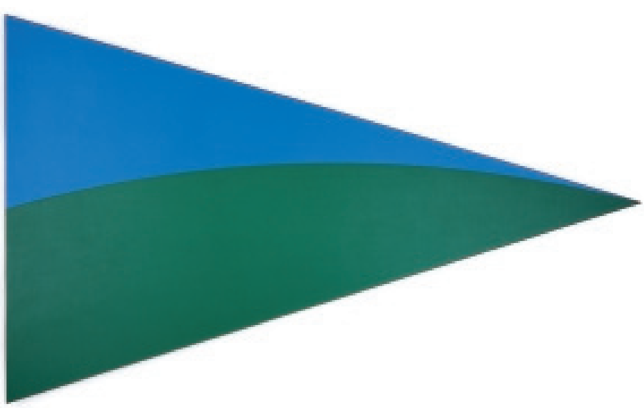
d.

A. increasing more and more rapidly

B. decreasing more and more slowly

C. increasing more and more slowly

D. decreasing more and more rapidly

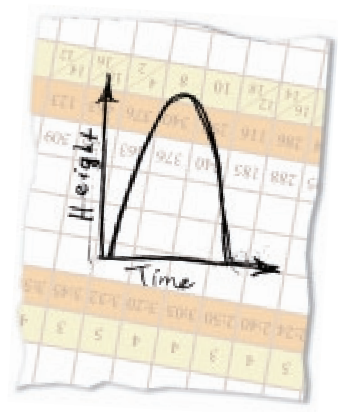


American minimalist painter and sculptor Ellsworth Kelly (b 1923) based many of his works on the shapes of shadows and spaces between objects.

Ellsworth Kelly *Blue Green Curve,* 1972, oil on canvas, 87-3/4 x 144-1/4 in. The Museum of Contemporary Art, Los Angeles, The Barry Lowen Collection

Reason and Apply

4. Harold's concentration often wanders from the game of golf to the mathematics involved in his game. His scorecard frequently contains mathematical doodles and graphs.



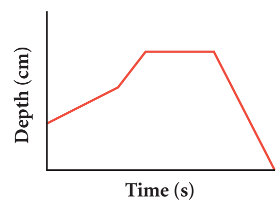
a. What is a real-world meaning for this graph found on one of his recent scorecards?

b. What units might he be using?

c. Describe a realistic domain and range for this graph.

d. Does this graph show how far the ball traveled? Explain.

5. Make up a story to go with the graph at right. Be sure to interpret the x- and y-intercepts.



6. Sketch what you think is a reasonable graph for each relationship described. In each situation, identify the variables and label your axes appropriately.

a. the height of a basketball during the last 10 seconds of a game

b. the distance it takes to brake a car to a full stop, compared to the car's speed when the brakes are first applied

c. the temperature of an iced drink as it sits on a table for a long period of time

d. the speed of a falling acorn after a squirrel drops it from the top of an oak tree

e. your height above the ground as you ride a Ferris wheel

7. Sketch what you think is a reasonable graph for each relationship described. In each situation, identify the variables and label your axes appropriately. In each situation, will the graph be continuous or will it be a collection of discrete points or pieces? Explain why.

a. the amount of money you have in a savings account that is compounded annually, over a period of several years, assuming no additional deposits are made

b. the same amount of money that you started with in 7a, hidden under your mattress over the same period of several years

c. an adult's shoe size compared to the adult's foot length

d. your distance from Detroit during a flight from Detroit to Newark if your plane is forced to circle the airport in a holding pattern when you approach Newark

e. the daily maximum temperature of a town, for a month

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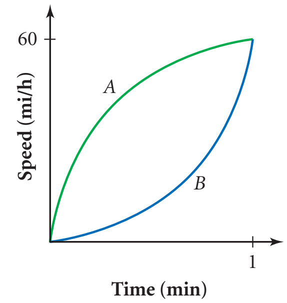
8. Describe a relationship of your own and draw a graph to go with it.

9. Car A and Car B are at the starting line of a race. At the green light, they both accelerate to 60 mi/h in 1 min. The graph at right represents their velocities in relation to time.

   a. Describe the rate of change for each car.

   b. After 1 minute, which car will be in the lead? Explain your reasoning.



Review

10. Write an equation for the line that fits each situation.

    a. The length of a rope is 1.70 m, and it decreases by 0.12 m for every knot that is tied in it.

    b. When you join a CD club, you get the first 8 CDs for \$7.00. After that, your bill increases by \$9.50 for each additional CD you purchase.

11. **APPLICATION** Albert starts a business reproducing high-quality copies of pictures. It costs \$155 to prepare the picture and then \$15 to make each print. Albert plans to sell each print for \$27.

    a. Write a cost equation and graph it.

    b. Write an income equation and graph it on the same set of axes.

    c. How many pictures does Albert need to sell before he makes a profit?

American photographer Gordon Parks (b 1912) holds a large, framed print of one of his photographs.

12. **APPLICATION** Suppose you have a \$200,000 home loan with an annual interest rate of 6.5%, compounded monthly.

    a. If you pay \$1200 per month, what balance remains after 20 years?

    b. If you pay \$1400 per month, what balance remains after 20 years?

    c. If you pay \$1500 per month, what balance remains after 20 years?

    d. Make an observation about the answers to 12a–c.

13. Follow these steps to solve this system of three equations in three variables.

$$\begin{cases} 2x + 3y - 4z = -9 & \text{(Equation 1)} \\ x + 2y + 4z = 0 & \text{(Equation 2)} \\ 2x - 3y + 2z = 15 & \text{(Equation 3)} \end{cases}$$

    a. Use the elimination method with Equation 1 and Equation 2 to eliminate z. The result will be an equation in two variables, x and y.

    b. Use the elimination method with Equation 1 and Equation 3 to eliminate z.

    c. Use your equations from 13a and b to solve for both x and y.

    d. Substitute the values from 13c into one of the original equations and solve for z. What is the solution to the system?

LESSON

4.2

Function Notation

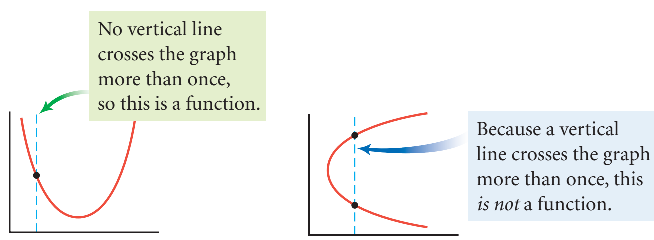
She had not understood mathematics until he had explained to her that it was the symbolic language of relationships. "And relationships," he had told her, "contained the essential meaning of life."

PEARL S. BUCK
THE GODDESS ABIDES, 1972

Rachel's parents keep track of her height as she gets older. They plot these values on a graph and connect the points with a smooth curve. For every age you choose on the x-axis, there is only one height that pairs with it on the y-axis. That is, Rachel is only one height at any specific time during her life.

A **relation** is any relationship between two variables. A **function** is a relationship between two variables such that for every value of the independent variable, there is at most one value of the dependent variable. A function is a special type of relation. If x is your independent variable, a function pairs at most one y with each x. You can say that Rachel's height is a function of her age.

You may remember the vertical line test from previous mathematics classes. It helps you determine whether or not a graph represents a function. If no vertical line crosses the graph more than once, then the relation is a function. Take a minute to think about how you could apply this technique to the graph of Rachel's height and the graph in the next example.



Function

Not a function

Function notation emphasizes the dependent relationship between the variables that are used in a function. The notation $y = f(x)$ indicates that values of the dependent variable, y, are explicitly defined in terms of the independent variable, x, by the function f. You read $y = f(x)$ as "y equals f of x."

Graphs of functions and relations can be continuous, such as the graph of Rachel's height, or they can be made up of discrete points, such as a graph of the maximum temperatures for each day of a month. Although real-world data often have an identifiable pattern, a function does not necessarily need to have a rule that connects the two variables.

Technology CONNECTION

A computer's desktop represents a function. Each icon, when clicked on, opens only one file, folder, or application.

Keymath.com Links to Resources
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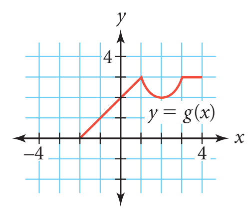
This handwritten music manuscript by Norwegian composer Edvard Grieg (1843–1907) shows an example of functional relationships. Each of the four simultaneous voices for which this hymn is written can sing only one note at a time, so for each voice the pitch is a function of time.

EXAMPLE

Function f is defined by the equation $f(x) = \frac{2x + 5}{x - 3}$.
Function g is defined by the graph at right.



Find these values.

a. $f(8)$

b. $f(-7)$

c. $g(1)$

d. $g(-2)$

▶ Solution

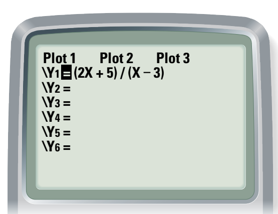
When a function is defined by an equation, you simply replace each x with the x-value and evaluate.

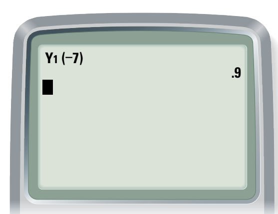
a. $f(x) = \dfrac{2x + 5}{x - 3}$

$$f(8) = \frac{2 \cdot 8 + 5}{8 - 3} = \frac{21}{5} = 4.2$$

b. $f(-7) = \dfrac{2 \cdot (-7) + 5}{-7 - 3} = \dfrac{-9}{-10} = 0.9$

You can check your work with your calculator. [▶ See **Calculator Note 4A** to learn about evaluating functions. ◀]





c. The notation $y = g(x)$ tells you that the values of y are explicitly defined, in terms of x, by the graph of the function g. To find $g(1)$, locate the value of y when x is 1. The point $(1, 3)$ on the graph means that $g(1) = 3$.

d. The point $(-2, 0)$ on the graph means that $g(-2) = 0$.

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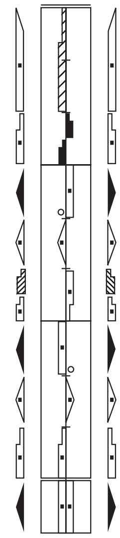
Award-winning tap dancers Gregory Hines (b 1946) and Savion Glover (b 1973) perform at the 2001 New York City Tap Festival.

At far right is Labanotation, a way of graphically representing dance. A single symbol shows you the direction, level, length of time, and part of the body performing a movement. This is a type of function notation because each part of the body can perform only one motion at any given time. For more information on dance notation, see the links at www.keymath.com/DAA .

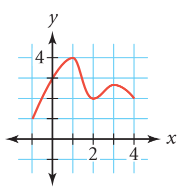


In the investigation you will practice identifying functions and using function notation. As you do so, notice how you can identify functions in different forms.

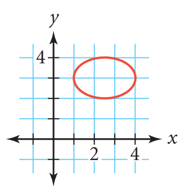
Investigation
To Be or Not to Be (a Function)

Below are nine representations of relations.

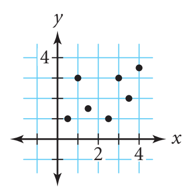
a.



b.



c.



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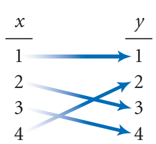
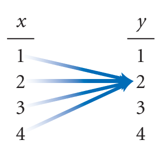
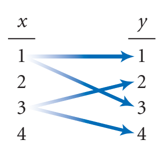


e.



f.



g. independent variable: the age of each student in your class
dependent variable: the height of each student

h. independent variable: an automobile in the state of Kentucky
dependent variable: that automobile's license plate number

i. independent variable: the day of the year
dependent variable: the time of sunset

Step 1 Identify each relation that is also a function. For each relation that is not a function, explain why not.

Step 2 For each function in parts a–f, find the y-value when $x = 2$, and find the x-value(s) when $y = 3$. Write each answer in function notation using the letter of the subpart as the function name, for example, $y = d(x)$ for part d.

When you use function notation to refer to a function, you can use any letter you like. For example, you might use $y = h(x)$ if the function represents height, or $y = p(x)$ if the function represents population. Often in describing real-world situations, you use a letter that makes sense. However, to avoid confusion, you should avoid using the dependent variable as the function name, as in $y = y(x)$. Choose freely but choose wisely.

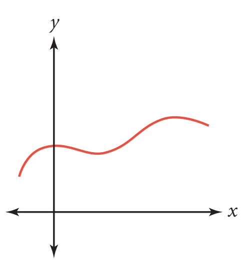
When looking at real-world data, it is often hard to decide whether or not there is a functional relationship. For example, if you measure the height of every student in your class and the weight of his or her backpack, you may collect a data set in which each student height is paired with only one backpack weight. But does that mean no two students of the same height could have backpacks of equal weight? Does it mean you shouldn't try to model the situation with a function?

EXERCISES

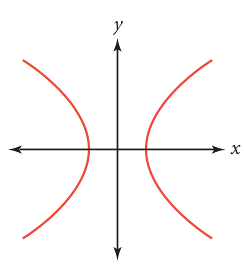
Practice Your Skills

1. Which of these graphs represent functions? Why or why not?

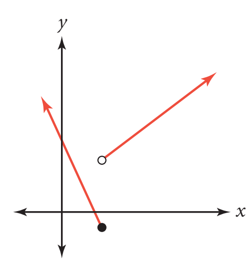
a.



b.



c.



2. Use the functions $f(x) = 3x - 4$ and $g(x) = x^2 + 2$ to find these values.

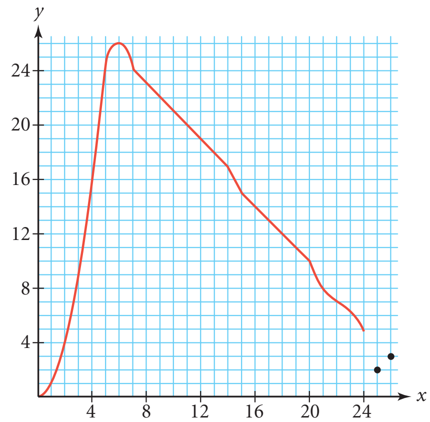
a. $f(7)$ **b.** $g(5)$ **c.** $f(-5)$ **d.** $g(-3)$ **e.** x when $f(x) = 7$

3. Miguel works at an appliance store. He gets paid \$5.25 an hour and works 8 hours a day. In addition, he earns a 3% commission on all items he sells. Let x represent the total dollar value of the appliances that Miguel sells, and let the function m represent Miguel's daily earnings as a function of x. Which function describes how much Miguel earns in a day?

A. $m(x) = 5.25 + 0.03x$ **B.** $m(x) = 42 + 0.03x$

C. $m(x) = 5.25 + 3x$ **D.** $m(x) = 42 + 3x$

4. Use the graph at right to find each value. Each answer will be an integer from 1 to 26. Relate each answer to a letter of the alphabet (1 = A, 2 = B, and so on), and fill in the name of a famous mathematician.



a. $f(13)$

b. $f(25) + f(26)$

c. $2f(22)$

d. $\dfrac{f(3) + 11}{\sqrt{f(3 + 1)}}$

e. $\dfrac{f(1 + 4)}{f(1) + 4} - \dfrac{1}{4}\left(\dfrac{4}{f(1)}\right)$

f. x when $f(x + 1) = 26$

g. $\sqrt[3]{f(21)} + f(14)$

h. x when $2f(x + 3) = 52$

i. x when $f(2x) = 4$

j. $f(f(2) + f(3))$

k. $f(9) - f(25)$

l. $f(f(5) - f(1))$

m. $f(4 \cdot 6) + f(4 \cdot 4)$

___ ___ ___ ___   ___ ___ ___ ___ ___ ___ ___ ___ ___

a b c d   **e f g h i j k l m**

5. Identify the independent variable for each relation. Is the relation a function?

a. the price of a graphing calculator and the sales tax you pay

b. the amount of money in your savings account and the time it has been in the account

c. the amount your hair has grown since the time of your last haircut

d. the amount of gasoline in your car's fuel tank and how far you have driven since your last fill-up

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Reason and Apply

6. Sketch a reasonable graph for each relation described in Exercise 5. In each situation, identify the variables and label your axes appropriately.

7. Suppose $f(x) = 25 - 0.6x$.

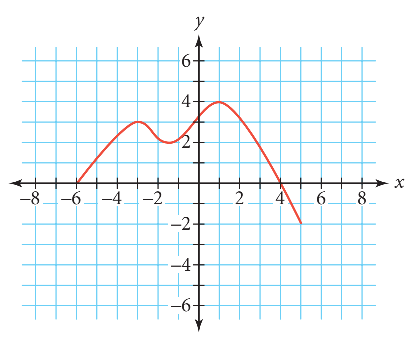
a. Draw a graph of this function.

b. What is $f(7)$?

c. Identify the point $(7, f(7))$ by marking it on your graph.

d. Find the value of x when $f(x) = 27.4$. Mark this point on your graph.

8. Identify the domain and range of the function of f in the graph at right.



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9. Sketch a graph for each function.

a. $y = f(x)$ has domain all real numbers and range $f(x) \leq 0$.

b. $y = g(x)$ has domain $x > 0$ and range all real numbers.

c. $y = h(x)$ has domain all real numbers and range $h(x) = 3$.

10. Consider the function $f(x) = 3(x + 1)^2 - 4$.

a. Find $f(5)$. **b.** Find $f(n)$. **c.** Find $f(x + 2)$.

d. Use your calculator to graph $y = f(x)$ and $y = f(x + 2)$ on the same axes. How do the graphs compare?

11. Kendall walks toward and away from a motion sensor. Is the graph of his motion a function? Why or why not?

12. **APPLICATION** The length of a pendulum in inches, L, is a function of its period, or the length of time it takes to swing back and forth, in seconds, t. The function is defined by the formula $L = 9.73t^2$.

a. Find the length of a pendulum if its period is 4 s.

b. The Foucault pendulum at the Panthéon in Paris has a 62-pound iron ball suspended on a 220-foot wire. What is its period?

Astronomer Jean Bernard Leon Foucault (1819–1868) displayed this pendulum for the first time in 1851. The floor underneath the swinging pendulum was covered in sand, and a pin attached to the ball traced out the pendulum's path. While the ball swung back and forth in nine straight lines, it changed direction relative to the floor, proving that the Earth was rotating underneath it.

13. The number of diagonals of a polygon, d, is a function of the number of sides of the polygon, n, and is given by the formula $d = \frac{n(n-3)}{2}$.

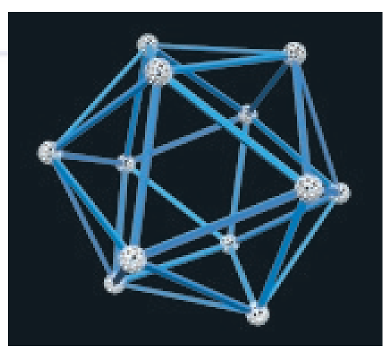
a. Find the number of diagonals in a dodecagon (a 12-sided polygon).

b. How many sides would a polygon have if it contained 170 diagonals?

Language CONNECTION

You probably have noticed that some words, like biannual, triplex, and quadrant, have prefixes that indicate a number. Knowing the meaning of a prefix can help you determine the meaning of a word. The word "polygon" comes from the Greek *poly-* (many) and *-gon* (angle). Many mathematical words use the following Greek prefixes.

1 mono	6 hexa	
2 di	7 hepta	
3 tri	8 octa	
4 tetra	9 ennea	
5 penta	10 deca	20 icosa



A polyhedron is a three-dimensional shape with many sides. Can you guess what the name of this shape is, using the prefixes given?

Review

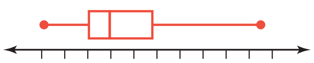
14. Create graphs picturing the water height as each bottle is filled with water at a constant rate.

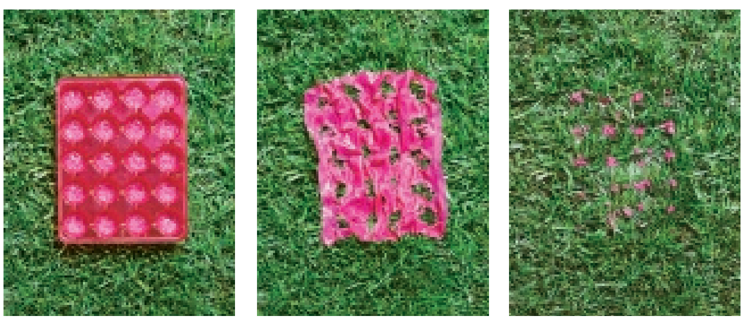
a.

b.

c.

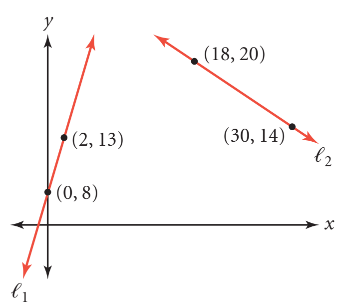
15. **APPLICATION** The five-number summary of this box plot is \$2.10, \$4.05, \$4.95, \$6.80, \$11.50. The plot summarizes the amounts of money earned in a recycling fund drive by 32 members of the Oakley High School environmental club. Estimate the total amount of money raised. Explain your reasoning.





These photos show the breakdown of a newly developed plastic during a one-hour period. Created by Australian scientists, the plastic is made of cornstarch and disintegrates rapidly when exposed to water. This technology could help eliminate the 24 million tons of plastic that end up in American landfills every year.

16. Given the graph at right, find the intersection of lines ℓ_1 and ℓ_2.



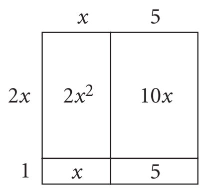
17. Sketch a graph for a function that has the following characteristics.

a. domain: $x \geq 0$
range: $f(x) \geq 0$
linear and increasing

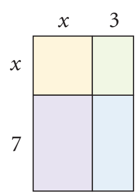
b. domain: $-10 \leq x \leq 10$
range: $-3 < f(x) \leq 3$
nonlinear and increasing

c. domain: $x \geq 0$
range: $-2 < f(x) \leq 10$
increasing, then decreasing, then increasing, and then decreasing

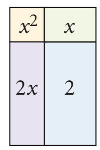
18. You can use rectangle diagrams to represent algebraic expressions. For instance, this diagram demonstrates the equation $(x + 5)(2x + 1) = 2x^2 + 11x + 5$. Fill in the missing values on the edges or in the interior of each rectangle diagram.



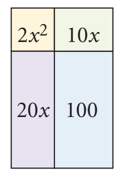
a.



b.



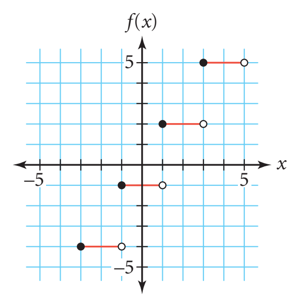
c.



Project

STEP FUNCTIONS

The graph at right is an example of a **step function.** The open circles mean that those points are not included in the graph. For example, the value of $f(3)$ is 5, not 2. The places where the graph “jumps” are called **discontinuities.**



In Lesson 3.6, Exercise 9, you were introduced to an often-used step function—the **greatest integer function,** $f(x) = [x]$. Two related functions are the ceiling function, $f(x) = \lceil x \rceil$, and the floor function, $f(x) = \lfloor x \rfloor$.

Do further research on the greatest integer function, the ceiling function, and the floor function. Prepare a report or class presentation on the functions. Your project should include

- A graph of each function.
- A written or verbal description of how each function operates, including any relationships among the three functions. Be sure to explain how you would evaluate each function for different values of x.
- Examples of how each function might be applied in a real-world situation.

As you do your research, you might learn about other step functions that you’d like to include in your project.

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LESSON 4.3

Keymath.com Links to Resources

Lines in Motion

In Chapter 3, you worked with two forms of linear equations:

Intercept form $y = a + bx$

Point-slope form $y = y_1 + b(x - x_1)$

In this lesson you will see how these forms are related to each other graphically.

With the exception of vertical lines, lines are functions. That means you could write the forms above as $f(x) = a + bx$ and $f(x) = f(x_1) + b(x - x_1)$. Linear functions are some of the simplest functions.

The investigation will help you see the effect that moving the graph of a line has on its equation. Moving a graph horizontally or vertically is called a **translation**. The discoveries you make about translations of lines will also apply to the graphs of other functions.

Free Basin (2002), shown here at the Wexner Center for the Arts in Columbus, Ohio, is a functional sculpture designed by Simparch, an artists' collaborative in Chicago, Illinois. As former skateboarders, the makers of *Free Basin* wanted to create a piece formed like a kidney-shaped swimming pool, to pay tribute to the empty swimming pools that first inspired skateboarding on curved surfaces. The underside of the basin shows beams that lie on lines that are translations of each other.

Investigation
Movin' Around

You will need

- two motion sensors

In this investigation you will explore what happens to the equation of a linear function when you translate the graph of the line. You'll then use your discoveries to interpret data.

Graph the lines in each step and look for patterns.

Step 1 On graph paper, graph the line $y = 2x$ and then draw a line parallel to it, but 3 units higher. What is the equation of this new line?

Step 2 On the same set of axes, draw a line parallel to the line $y = 2x$, but shifted down 4 units. What is the equation of this line?

Step 3 On a new set of axes, graph the line $y = \frac{1}{2}x$. Mark the point where the line passes through the origin. Plot another point right 3 units and up 4 units from the origin, and draw a line through this point parallel to the original line. Write at least two equations of the new line.

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Step 4 What happens if you move every point on $f(x) = \frac{1}{2}x$ to a new point up 1 unit and right 2 units? Write an equation in point-slope form for this new line. Then distribute and combine like terms to write the equation in intercept form. What do you notice?

Step 5 In general, what effect does translating a line have on its equation?

Your group will now use motion sensors to create a function and a translated copy of that function. [▶ See **Calculator Note 4B** for instructions on how to collect and retrieve data from two motion sensors. ◀]

Step 6 Arrange your group as in the photo to collect data.

Step 7 Person D coordinates the collection of data like this:

At 0 seconds:	C begins to walk slowly toward the motion sensors, and A begins to collect data.
About 2 seconds:	B begins to collect data.
About 5 seconds:	C begins to walk backward.
About 10 seconds:	A's sensor stops.
About 12 seconds:	B's sensor stops and C stops walking.

Step 8 After collecting the data, follow Calculator Note 4B to retrieve the data to two calculators and then transmit four lists of data to each group member's calculator. Be sure to keep track of which data each list contains.

Step 9 Graph both sets of data on the same screen. Record a sketch of what you see and answer these questions:

a. How are the two graphs related to each other?

b. If A's graph is $y = f(x)$, what equation describes B's graph? Describe how you determined this equation.

c. In general, if the graph of $y = f(x)$ is translated horizontally h units and vertically k units, what is the equation of this translated function?

If you know the effects of translations, you can write an equation that translates any function on a graph. No matter what the shape of a function $y = f(x)$ is, the graph of $y = f(x - 3) + 2$ will look just the same as $y = f(x)$, but it will be translated up 2 units and right 3 units. Understanding this relationship will enable you to graph functions and write equations for graphs more easily.

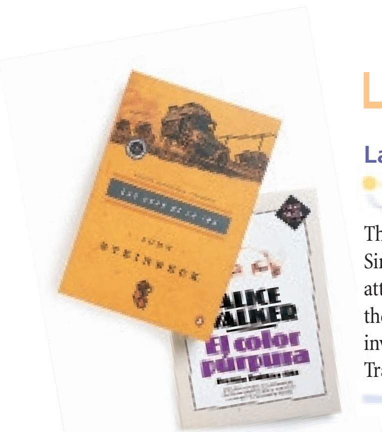
Translation of a Function

A **translation** moves a graph horizontally, or vertically, or both.

Given the graph of $y = f(x)$, the graph of

$$y = f(x - h) + k$$

is a translation horizontally h units and vertically k units.



Pulitzer Prize–winning books *The Color Purple,* written in 1982 by Alice Walker (b 1944), and *The Grapes of Wrath,* written in 1939 by John Steinbeck (1902–1968), are shown here in Spanish translations.

Language CONNECTION

The word "translation" can refer to the act of converting between two languages. Similar to its usage in mathematics, *translation* of foreign languages is an attempt to keep meanings parallel. Direct substitution of words often destroys the nuances and subtleties of meaning of the original text. The subtleties involved in the art and craft of translation have inspired the formation of Translation Studies programs in universities throughout the world.

In a translation, every point (x_1, y_1) is mapped to a new point, $(x_1 + h, y_1 + k)$. This new point is called the **image** of the original point. If you have difficulty remembering which way to move a function, think about the point-slope form of the equation of a line. In $y = y_1 + b(x - x_1)$, the point at (0, 0) is translated to the new point at (x_1, y_1). In fact, every point is translated horizontally x_1 units and vertically y_1 units.

Panamanian cuna (mola with geometric design on red background)

EXAMPLE

Describe how the graph of $f(x) = 4 + 2(x - 3)$ is a translation of the graph of $f(x) = 2x$.

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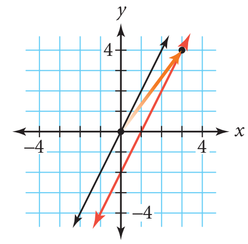
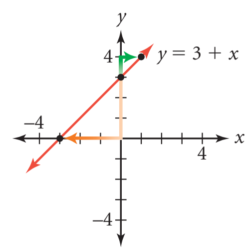
▶ Solution

The graph of $f(x) = 4 + 2(x - 3)$ passes through the point (3, 4). Consider this point to be the translated image of (0, 0) on $f(x) = 2x$. It is translated right 3 units and up 4 units from its original location, so the graph of $f(x) = 4 + 2(x - 3)$ is simply the graph of $f(x) = 2x$ translated right 3 units and up 4 units.

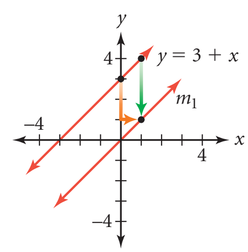


Note that you can distribute and combine like terms in $f(x) = 4 + 2(x - 3)$ to get $f(x) = -2 + 2x$. The fact that these two equations are equivalent means that translating the graph of $f(x) = 2x$ right 3 units and up 4 units is equivalent to translating the line down 2 units. In the graph in the example, this appears to be true.

If you imagine translating a line in a plane, there are some translations that will give you the same line you started with. For example, if you start with the line $y = 3 + x$ and translate every point up 1 unit and right 1 unit, you will map the line onto itself. If you translate every point on the line down 3 units and left 3 units, you also map the line onto itself.



There are infinitely many translations that map a line onto itself. Similarly, there are infinitely many translations that map a line onto another parallel line, m_1. To map the line $y = 3 + x$ onto the line m_1 shown below it, you could translate every point down 2 units and right 1 unit, or you could translate every point down 3 units.



In the next few lessons, you will see how to translate and otherwise transform other functions.

EXERCISES

Practice Your Skills

1. The graph of the line $y = \frac{2}{3}x$ is translated right 5 units and down 3 units. What is the equation of the new line?

2. How does the graph of $y = f(x - 3)$ compare with the graph of $y = f(x)$?

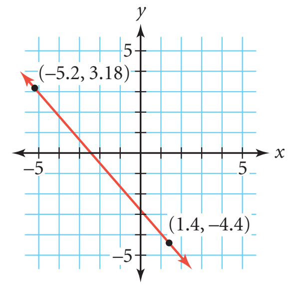
3. If $f(x) = -2x$, find

a. $f(x + 3)$ **b.** $-3 + f(x - 2)$ **c.** $5 + f(x + 1)$

4. Consider the line that passes through the points $(-5.2, 3.18)$ and $(1.4, -4.4)$, as shown.

a. Find an equation of the line.

b. Write an equation of the parallel line that is 2 units above this line.



5. Write an equation of each line.

a. the line $y = 4.7x$ translated down 3 units

b. the line $y = -2.8x$ translated right 2 units

c. the line $y = -x$ translated up 4 units and left 1.5 units

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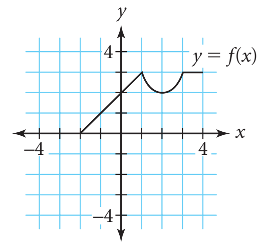
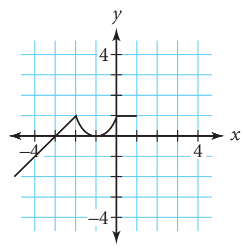
Reason and Apply

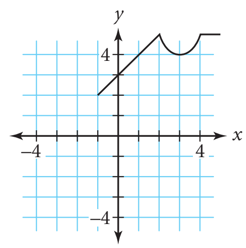
6. The graph of $y = f(x)$ is shown at right. Write an equation for each of the graphs below.



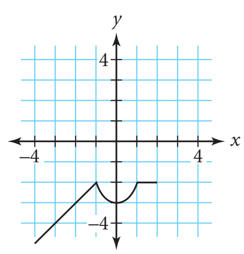
a.



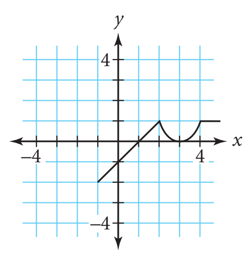
b.



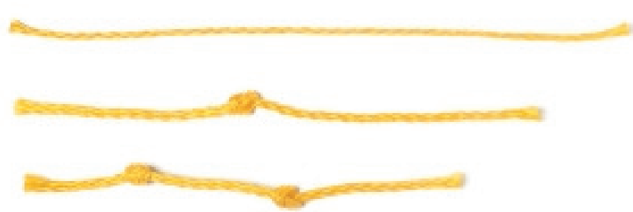
c.



d.



7. Jeannette and Keegan collect data about the length of a rope as knots are tied in it. The equation that fits their data is $y = 102 - 6.3x$, where x represents the number of knots and y represents the length of the rope in centimeters. Mitch had a piece of rope cut from the same source. Unfortunately he lost his data and can remember only that his rope was 47 cm long after he tied 3 knots. What equation describes Mitch's rope?



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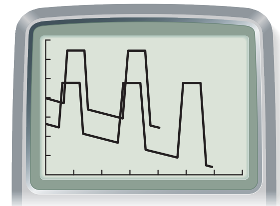
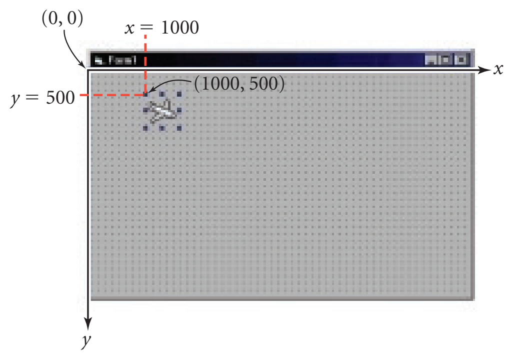
8. Rachel, Pete, and Brian perform Part 2 of the investigation in this lesson. Rachel walks while Pete and Brian hold the motion sensors. They create the unusual graph at right. The horizontal axis has a mark every 1 s, and the vertical axis has a mark every 1 m.



a. The lower curve is made from the data collected by Pete's motion sensor. Where was Brian standing and when did he start his motion sensor to create the upper curve?

b. If Pete's curve is the graph of $y = f(x)$, what equation represents Brian's curve?

9. **APPLICATION** Kari's assignment in her computer programming course is to simulate the motion of an airplane by repeatedly translating it across the screen. The coordinate system in the software program is shown at right with the origin, (0, 0), in the upper left corner. In this program, coordinates to the right and down are positive. The starting position of the airplane is (1000, 500), and Kari would like the airplane to end at (7000, 4000). She thinks that moving the airplane in 15 equal steps will model the motion well.



a. What should be the airplane's first position after (1000, 500)?

b. If the airplane's position at any time is given by (x, y), what is the next position in terms of x and y?

c. If the plane moves down 175 units and right 300 units in each step, how many steps will it take to reach the final position of (7000, 4000)?

Art CONNECTION

Animation simulates movement. An old-fashioned way to animate is to make a book of closely related pictures and flip the pages. Flipbook technique is used in cartooning—a feature-length film might have more than 65,000 images. Today, this tedious hand drawing has been largely replaced by computer-generated special effects.

Since the mid-1990s Macromedia Flash animation has given websites striking visual effects.

© 2002 Eun-Ha Paek. Stills from "L'Faux Episode 7" on www.MilkyElephant.com

10. ***Mini-Investigation*** Linear equations can also be written in standard form.

Standard form $\quad ax + by = c$

a. Identify the values of a, b, and c for each of these equations in standard form.

i. $4x + 3y = 12$	**ii.** $-x + y = 5$
iii. $7x - y = 1$	**iv.** $-2x + 4y = -2$
v. $2y = 10$	**vi.** $3x = -6$

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b. Solve the standard form, $ax + by = c$, for y. The result should be an equivalent equation in intercept form. What is the y-intercept? What is the slope?

c. Use what you've learned from 10b to find the y-intercept and slope of each of the equations in 10a.

d. The graph of $4x + 3y = 12$ is translated as described below. Write an equation in standard form for each of the translated graphs.

i. a translation right 2 units

ii. a translation left 5 units

iii. a translation up 4 units

iv. a translation down 1 unit

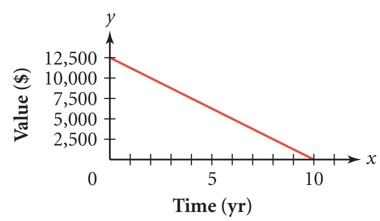
v. a translation right 1 unit and down 3 units

vi. a translation up 2 units and left 2 units

e. In general, if the graph of $ax + by = c$ is translated horizontally h units and vertically k units, what is the equation of the translated line?

Review

11. **APPLICATION** The Internal Revenue Service has approved ten-year linear depreciation as one method for determining the value of business property. This means that the value declines to zero over a ten-year period, and you can claim a tax exemption in the amount of the value lost each year. Suppose a piece of business equipment costs \$12,500 and is depreciated over a ten-year period. At right is a sketch of the linear function that represents this depreciation.



a. What is the y-intercept? Give the real-world meaning of this value.

b. What is the x-intercept? Give the real-world meaning of this value.

c. What is the slope? Give the real-world meaning of the slope.

d. Write an equation that describes the value of the equipment during the ten-year period.

e. When is the equipment worth \$6500?

12. Suppose that your basketball team's scores in the first four games of the season were 86 points, 73 points, 76 points, and 90 points.

a. What will be your team's mean score if the fifth-game score is 79 points?

b. Write a function that gives the mean score in terms of the fifth-game score.

c. What score will give a five-game average of 84 points?

13. Solve.

a. $2(x + 4) = 38$

b. $7 + 0.5(x - 3) = 21$

c. $-2 + \frac{3}{4}(x + 1) = -17$

d. $4.7 + 2.8(x - 5.1) = 39.7$

14. The three summary points for a data set are $M_1(3, 11)$, $M_2(5, 5)$, and $M_3(9, 2)$. Find the median-median line.

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LESSON 4.4

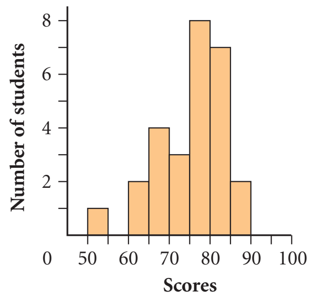
Translations and the Quadratic Family

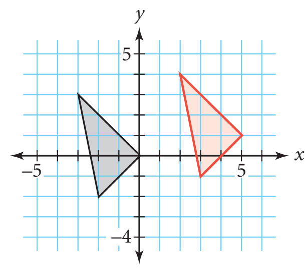
I see music as the augmentation of a split second of time.

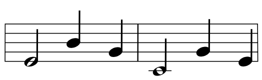
ERIN CLEARY

In the previous lesson, you looked at translations of the graphs of linear functions. Translations can occur in other settings as well. For instance, what will this histogram look like if the teacher decides to add five points to each of the scores?

What translation will map the black triangle on the left onto its red image on the right?





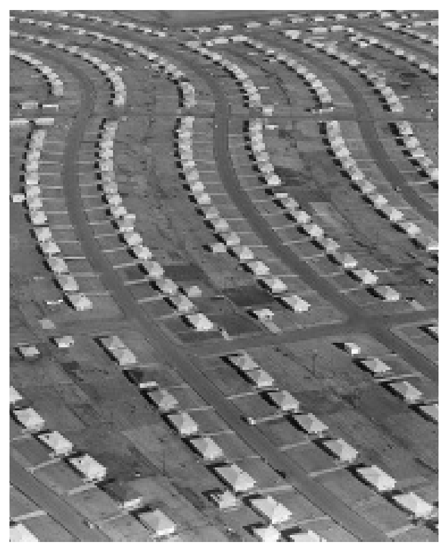


Music CONNECTION

When a song is in a key that is difficult to sing or play, it can be translated, or transposed, into an easier key. To transpose music means to change the pitch of each note without changing the relationships between the notes. Musicians have several techniques for transposing music, and because these techniques are mathematically based, computer programs have been written that can do it as well.

Translations are also a natural feature of the real world, including the world of art. Music can be transposed from one key to another. Melodies are often translated by a certain interval within a composition.

Jazz saxophonist Ornette Coleman (b 1930) grew up with strong interests in mathematics and science. Since the 1950s, he has developed award-winning musical theories, such as "free jazz," which strays from the set standards of harmony and melody.



This suburb of St. Paul, Minnesota, was developed in the 1950s. A close look reveals that some of the houses are translations of each other. A few are reflections of each other.

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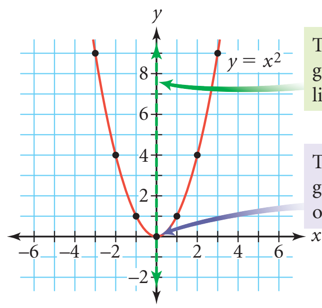
In mathematics, a change in the size or position of a figure or graph is called a **transformation.** Translations are one type of transformation. You may recall other types of transformations, such as reflections, dilations, stretches, shrinks, and rotations, from other mathematics classes.

In this lesson you will experiment with translations of the graph of the function $y = x^2$. The special shape of this graph is called a **parabola.** Parabolas always have a **line of symmetry** that passes through the **vertex.**



The line of symmetry divides the graph into mirror-image halves. The line of symmetry of $y = x^2$ is $x = 0$.

The vertex is the point where the graph changes direction. The vertex of $y = x^2$ is $(0, 0)$.

The function $y = x^2$ is a building-block function, or **parent function.** By transforming the graph of a parent function, you can create infinitely many new functions, or a **family of functions.** The function $y = x^2$ and all functions created from transformations of its graph are called **quadratic functions,** because the highest power of x is x-squared.

Quadratic functions are very useful, as you will discover throughout this book. You can use functions in the quadratic family to model the height of a projectile as a function of time, or the area of a square as a function of the length of its side.

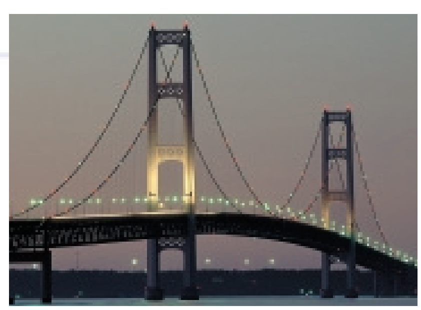
The focus of this lesson is on writing the quadratic equation of a parabola after a translation and graphing a parabola given its equation. You will see that locating the vertex is fundamental to your success with understanding parabolas.

Bessie's Blues, by American artist Faith Ringgold (b 1930), shows 25 stenciled images of blues artist Bessie Smith. Was the stencil translated or reflected to make each image? How can you tell?

Bessie's Blues, by Faith Ringgold ©1997, acrylic on canvas, 76″ × 79.″ Photo courtesy of the artist.

Engineering CONNECTION

Several types of bridge designs involve the use of curves modeled by nonlinear functions. Each main cable of a suspension bridge approximates a parabola. To learn more about the design and construction of bridges, see the links at www.keymath.com/DAA .



The five-mile long Mackinac Bridge in Michigan was built in 1957.

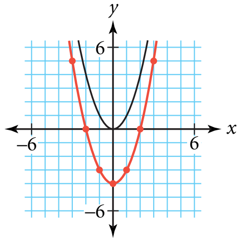
Investigation
Make My Graph

Procedure Note

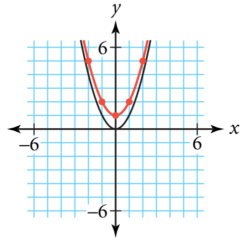
For this investigation, use a "friendly" calculator window with a factor of 2. [▸ See **Calculator Note 4C** to learn about friendly windows. ◂] Enter the parent function $y = x^2$ into Y_1. Enter the equation for the transformation in Y_2, and graph both Y_1 and Y_2 to check your work.

Step 1 Each graph below shows the graph of the parent function $y = x^2$ in black. Find a quadratic equation that produces the congruent, red parabola. Apply what you learned about translations of the graphs of linear equations in Lesson 4.3.

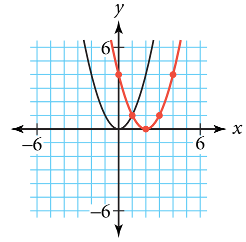
a.



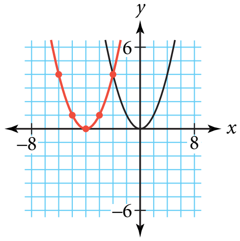
b.



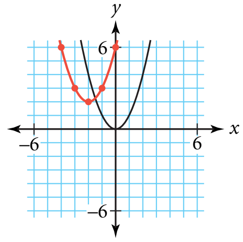
c.



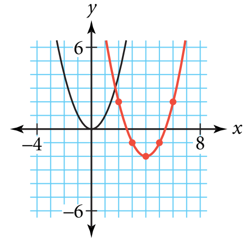
d.



e.



f.



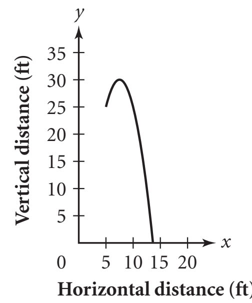
Step 2 Write a few sentences describing any connections you discovered between the graphs of the translated parabolas, the equation for the translated parabola, and the equation of the parent function $y = x^2$.

Step 3 In general, what is the equation of the parabola formed when the graph of $y = x^2$ is translated horizontally h units and vertically k units?

The following example shows one simple application involving parabolas and translations of parabolas. In later chapters you will discover many applications of this important mathematical curve.

EXAMPLE

This graph shows a portion of a parabola. It represents a diver's position (horizontal and vertical distance) from the edge of a pool as he dives from a 5 ft long board 25 ft above the water.



a. Sketch a graph of the diver's position if he dives from a 10 ft long board 10 ft above the water. (Assume that he leaves the board at the same angle and with the same force.)

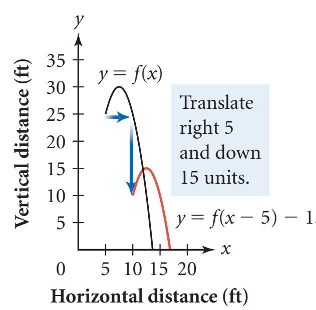
b. In the scenario described in part a, what is the diver's position when he reaches his maximum height?

▶ Solution

First, make sure that you can interpret the graph. The point (5, 25) represents the moment when the diver leaves the board, which is 5 ft long and 25 ft high. The vertex, (7.5, 30), represents the position where the diver's height is at a maximum, or 30 ft; it is also the point where the diver's motion changes from upward to downward. The x-intercept, approximately (13.6, 0), indicates that the diver hits the water at approximately 13.6 ft from the edge of the pool.

Mark Ruiz placed first in the 2000 U.S. Olympic Diving Team trials with this dive.

a. If the length of the board increases from 5 ft to 10 ft, then the parabola translates right 5 units. If the height of the board decreases from 25 ft to 10 ft, then the parabola translates down 15 units. If you define the original parabola as the graph of $y = f(x)$, then the function for the new graph is $y = f(x - 5) - 15$.



b. As with every point on the graph, the vertex translates right 5 units and down 15 units. The new vertex is $(7.5 + 5, 30 - 15)$, or $(12.5, 15)$. This means that when the diver's horizontal distance from the edge of the pool is 12.5 ft, he reaches his maximum height of 15 ft.

The translations you investigated with linear functions and functions in general work the same way with quadratic functions. If you translate the graph of $y = x^2$ horizontally h units and vertically k units, then the equation of the translated parabola is $y = (x - h)^2 + k$. You may also see this equation written as $y = k + (x - h)^2$ or $y - k = (x - h)^2$. When you translate any equation horizontally, you can think of it as replacing x in the equation with $(x - h)$. Likewise, a vertical translation replaces y with $(y - k)$.

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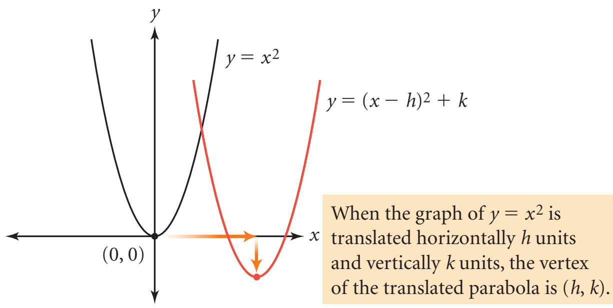
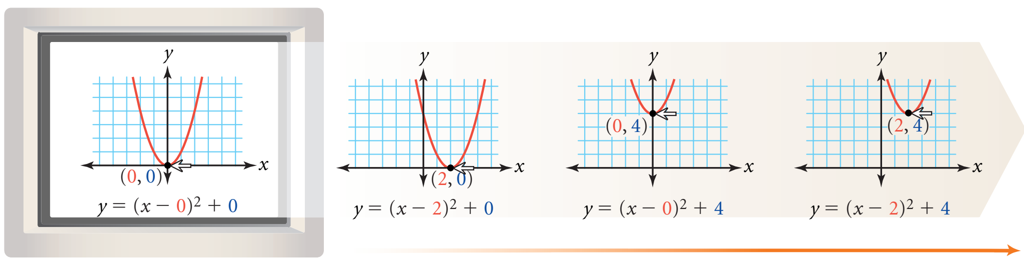
It is important to notice that the vertex of the translated parabola is (h, k). That's why finding the vertex is fundamental to determining translations of parabolas. In every function you learn, there will be key points to locate. Finding the relationships between these points and the corresponding points in the parent function enables you to write equations more easily.





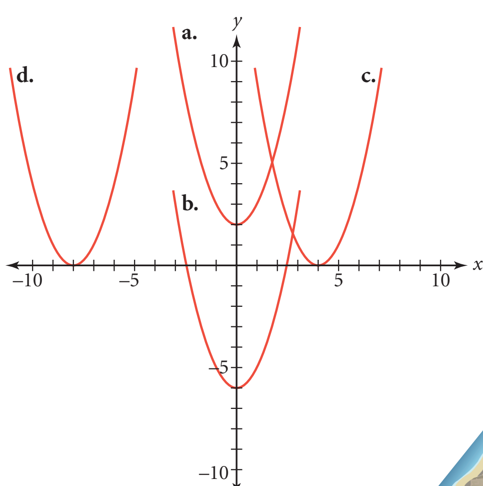
EXERCISES

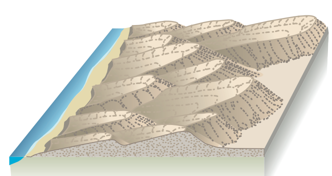
You will need

Geometry software for Exercises 15 and 16

Practice Your Skills

1. Write an equation for each parabola. Each parabola is a translation of the graph of the parent function $y = x^2$.





These black sand dunes in the Canary Islands, off the coast of Africa, form parabolic shapes called deflation hollows.

2. Each parabola described is congruent to the graph of $y = x^2$. Write an equation for each parabola and sketch its graph.

a. The parabola is translated down 5 units.

b. The parabola is translated up 3 units.

c. The parabola is translated right 3 units.

d. The parabola is translated left 4 units.

3. If $f(x) = x^2$, then the graph of each equation below is a parabola. Describe the location of the parabola relative to the graph of $f(x) = x^2$.

a. $y = f(x) - 3$

b. $y = f(x) + 4$

c. $y = f(x - 2)$

d. $y = f(x + 4)$

4. Describe what happens to the graph of $y = x^2$ in the following situations.

a. x is replaced with $(x - 3)$.

b. x is replaced with $(x + 3)$.

c. y is replaced with $(y - 2)$.

d. y is replaced with $(y + 2)$.

Keymath.com More Practice Your Skills

5. Solve.

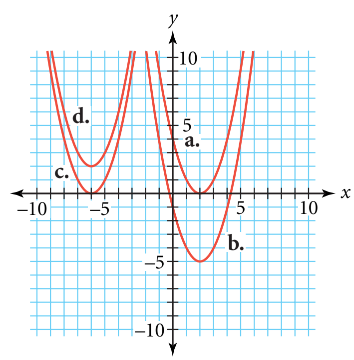
a. $x^2 = 4$

b. $x^2 + 3 = 19$

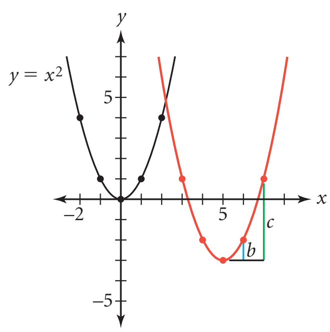
c. $(x - 2)^2 = 25$

Reason and Apply

6. Write an equation for each parabola at right.



7. The red parabola below is the image of the graph of $y = x^2$ after a translation right 5 units and down 3 units.

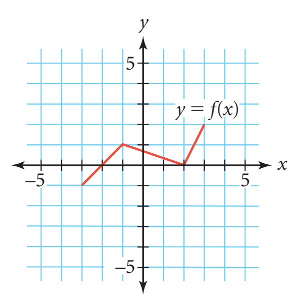


a. Write an equation for the red parabola.

b. Where is the vertex of the red parabola?

c. What are the coordinates of the other four points if they are 1 or 2 horizontal units from the vertex? How are the coordinates of each point on the black parabola related to the coordinates of the corresponding point on the red parabola?

d. What is the length of blue segment b? Of green segment c?



8. Given the graph of $y = f(x)$ at right, draw a graph of each of these related functions.

a. $y = f(x + 2)$

b. $y = f(x - 1) - 3$

9. **APPLICATION** This table of values compares the number of teams in a pee wee teeball league and the number of games required for each team to play every other team twice (once at home and once away from home).

Number of teams (x)	1	2	3	. . .
Number of games (y)	0	2	6	. . .

a. Continue the table out to 10 teams.

b. Plot each point and describe the graph produced.

c. Write an explicit function for this graph.

d. Use your function to find how many games are required if there are 30 teams.

10. Solve.

a. $3 + (x - 5)^2 = 19$

b. $(x + 3)^2 = 49$

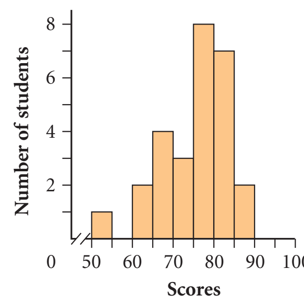
c. $5 - (x - 1) = -22$

d. $-15 + (x + 6)^2 = -7$

11. This histogram shows the students' scores on a recent quiz in Ms. Noah's class. Sketch what the histogram will look like if Ms. Noah

a. adds five points to everyone's score.

b. subtracts ten points from everyone's score.



Review

12. Match each recursive formula with the equation of the line that contains the sequence of points, (n, u_n), generated by the formula.

a. $u_0 = -8$
$u_n = u_{(n-1)} + 3$ where $n \geq 1$

b. $u_1 = 3$
$u_n = u_{(n-1)} - 8$ where $n \geq 2$

A. $y = 3x - 11$

B. $y = 3x - 8$

C. $y = 11 - 8x$

D. $y = -8x + 3$

13. **APPLICATION** You need to rent a car for one day. Mertz Rental charges \$32 per day plus \$0.10 per mile. Saver Rental charges \$24 per day plus \$0.18 per mile. Luxury Rental charges \$51 per day with unlimited mileage.

a. Write a cost equation for each rental agency.

b. Graph the three equations on the same axes.

c. Describe which rental agency is the cheapest alternative under various circumstances.

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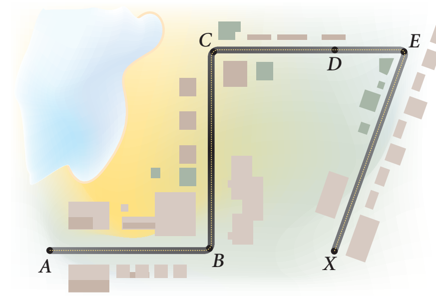
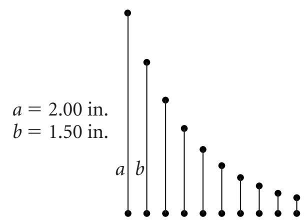
14. A car drives at a constant speed along the road pictured at right from point A to point X. Sketch a graph showing the straight line distance between the car and point X as it travels along the road. Mark points A, B, C, D, E, and X on your graph.



15. *Technology* Use geometry software to construct a segment whose length represents the starting term of a sequence. Then use transformations, such as translations and dilations, to create segments whose lengths represent additional terms in the sequence. For example, the segments at right represent the first ten terms of the sequence.

$u_1 = 2$

$u_n = 0.75 \cdot u_{n-1}$ where $n \geq 2$



16. *Technology* Use geometry software to investigate the form $y = ax + b$ of a linear function.

a. On the same coordinate plane, graph the lines $y = 0.5x + 4$, $y = x + 4$, $y = 2x + 4$, $y = 5x + 4$, $y = -3x + 4$, and $y = -0.25x + 4$. Describe the graphs of the family of lines $y = ax + 4$ as a takes on different values.

b. On the same coordinate plane, graph the lines $y = 2x - 7$, $y = 2x - 2$, $y = 2x$, $y = 2x + 3$, and $y = 2x + 8$. Describe the graphs of the family of lines $y = 2x + b$ as b takes on different values.

IMPROVING YOUR REASONING SKILLS

The Dipper

The group of stars known as the Big Dipper, which is part of the constellation Ursa Major, contains stars at various distances from Earth. Imagine translating the Big Dipper to a new position. Would all of the stars need to be moved the same distance? Why or why not?

Now imagine rotating the Big Dipper around the Earth. Do all the stars need to be moved the same distance? Why or why not?

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LESSON

4.5

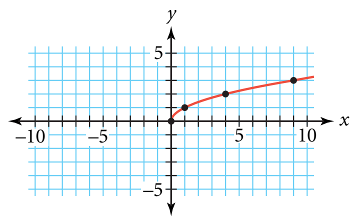
Keymath.com Links to Resources

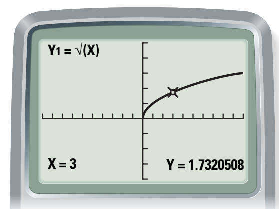
Call it a clan, call it a network, call it a tribe, call it a family. Whatever you call it, whoever you are, you need one.

JANE HOWARD

Reflections and the Square Root Family

The graph of the **square root function**, $y = \sqrt{x}$, is another parent function that you can use to illustrate transformations. From the graphs below, what are the domain and range of $f(x) = \sqrt{x}$? If you graph $y = \sqrt{x}$ on your calculator, you can trace to show that $\sqrt{3}$ is approximately 1.732. What is the approximate value of $\sqrt{8}$? How would you use the graph to find $\sqrt{31}$? What happens when you try to trace for values of $x < 0$?





$[-9.4, 9.4, 1, -6.2, 6.2, 1]$

Investigation

Take a Moment to Reflect

Procedure Note

For this investigation, use a friendly window with a factor of 2.

In this investigation you first will work with linear functions to discover how to create a new transformation—a **reflection**. Then you will apply reflections to quadratic functions and square root functions.

Step 1 Enter $y = 0.5x + 2$ into Y_1 and graph it on your calculator. Then enter the equation $Y_2 = -Y_1(x)$ and graph it.

a. Write the equation for Y_2 in terms of x. How does the graph of Y_2 compare with the graph of Y_1?

b. Change Y_1 to $y = -2x - 4$ and repeat the instructions in Step 1a.

c. Change Y_1 to $y = x^2 + 1$ and repeat.

d. In general, how are the graphs of $y = f(x)$ and $y = -f(x)$ related?

Step 2 Enter $y = 0.5x + 2$ into Y_1. Enter the equation $Y_2 = Y_1(-x)$ and graph both Y_1 and Y_2.

a. Write the equation for Y_2 in terms of x. How does the graph of Y_2 compare with the graph of Y_1?

b. Change Y_1 to $y = -2x - 4$ and repeat the instructions in Step 2a.

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c. Change Y_1 to $y = x^2 + 1$ and repeat. Explain what happens.

d. Change Y_1 to $y = (x - 3)^2 + 2$ and repeat.

e. In general, how are the graphs of $y = f(x)$ and $y = f(-x)$ related?

Step 3 Enter $y = \sqrt{x}$ into Y_1 and graph it on your calculator.

a. Predict what the graphs of $Y_2 = -Y_1(x)$ and $Y_2 = Y_1(-x)$ will look like. Use your calculator to verify your predictions. Write equations for both of these functions in terms of x.

b. Predict what the graph of $Y_2 = -Y_1(-x)$ will look like. Use your calculator to verify your prediction.

c. Do you notice that the graph of the square root function looks like half of a parabola, oriented horizontally? Why isn't it an entire parabola? What function would you graph to complete the bottom half of the parabola?

Reflections over the x- or y-axis are summarized below.

Reflection of a Function

A **reflection** is a transformation that flips a graph across a line, creating a mirror image.

Given the graph of $y = f(x)$,

the graph of $y = f(-x)$ is a reflection across the y-axis, and
the graph of $y = -f(x)$ is a reflection across the x-axis.

Because the graph of the square root function looks like half a parabola, it's easy to see the effects of reflections. The square root family has many real-world applications, such as finding the time it takes a falling object to reach the ground. The next example shows you how you can apply a square root function.

Science CONNECTION

Obsidian, a natural volcanic glass, was a popular material for tools and weapons in prehistoric times because it makes a very sharp edge. In 1960, scientists Irving Friedman and Robert L. Smith discovered that obsidian absorbs moisture at a slow, predictable rate and that measuring the thickness of the layer of moisture with a high-power microscope helps determine its age. Therefore, obsidian hydration dating can be used on obsidian artifacts, just as carbon dating can be used on organic remains. The age of prehistoric artifacts is predicted by a square root function similar to $d = \sqrt{5t}$ where t is time in thousands of years and d is the thickness of the layer of moisture in microns (millionths of a meter).

These flaked obsidian arrowheads—once used for cutting, carving, and hunting—were made by Native Americans near Jackson Lake, Wyoming more than 8500 years ago.

EXAMPLE

Objects fall to the ground because of the influence of gravity. When an object is dropped from an initial height of d meters, the height, h, after t seconds is given by the quadratic function $h = -4.9t^2 + d$. If an object is dropped from a height of 1000 meters, how long does it take for the object to fall to a height of 750 meters? 500 meters? How long will it take the object to hit the ground?

Science CONNECTION

English scientist Isaac Newton (1643–1727) formulated the theory of gravitation in the 1680s, building on the work of earlier scientists. Gravity is the force of attraction that exists between all objects. In general, larger objects pull smaller objects toward them. The force of gravity keeps objects on the surface of a planet, and it keeps objects in orbit around a planet or the Sun.

When an object falls near the surface of Earth, it speeds up, or accelerates. The acceleration caused by gravity is approximately 9.8 m/s^2, or 32 ft/s^2.

This time-lapse photograph by James Sugar shows an apple and feather falling at the same rate in a vacuum chamber. In the early 1600s, Galileo Galilei demonstrated that all objects fall at the same rate regardless of their weight, as long as they are not influenced by air resistance or other factors.

▸ Solution

The height of an object dropped from a height of 1000 meters is given by the function $h = -4.9t^2 + 1000$. You want to know t for various values of h, so first solve this equation for t.

$h = -4.9t^2 + 1000$	Original equation.
$h - 1000 = -4.9t^2$	Subtract 1000 from both sides.
$\dfrac{h - 1000}{-4.9} = t^2$	Divide by -4.9.
$\pm\sqrt{\dfrac{h - 1000}{-4.9}} = t$	Take the square root of both sides.
$t = \sqrt{\dfrac{h - 1000}{-4.9}}$	Because it doesn't make sense to have a negative value for time, use only the positive root.

To find when the height is 750 meters, substitute 750 for h.

$$t = \sqrt{\frac{750 - 1000}{-4.9}}$$

$$t \approx 7.143$$

The height of the object is 750 meters after approximately 7 seconds.

A similar substitution shows that the height of the object is 500 meters after approximately 10 seconds.

$$t = \sqrt{\frac{500 - 1000}{-4.9}} \approx 10.102$$

The object hits the ground when its height is 0 meters. That occurs after approximately 14 seconds.

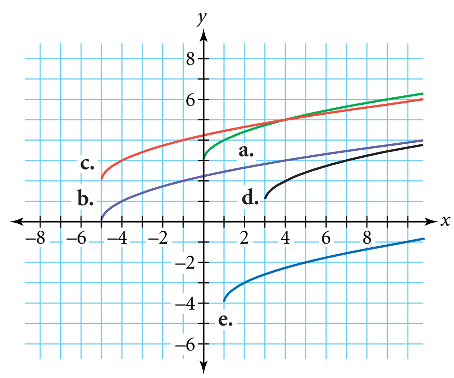
$$t = \sqrt{\frac{0 - 1000}{-4.9}} \approx 14.286$$

From the example, you may notice that square root functions play an important part in solving quadratic functions. Note that you cannot always eliminate the negative root as you did in the example. You'll have to let the context of a problem dictate when to use the positive root, the negative root, or both.

EXERCISES

Practice Your Skills

1. Each graph at right is a transformation of the graph of the parent function $y = \sqrt{x}$. Write an equation for each graph.



2. Describe what happens to the graph of $y = \sqrt{x}$ in the following situations.

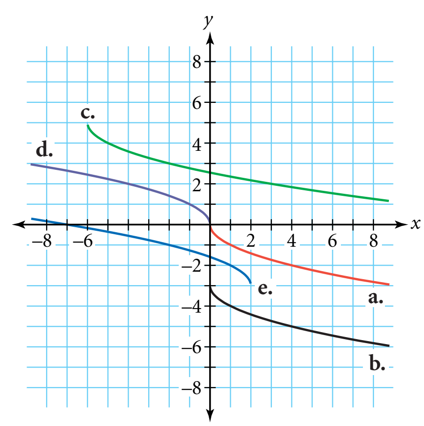
a. x is replaced with $(x - 3)$.

b. x is replaced with $(x + 3)$.

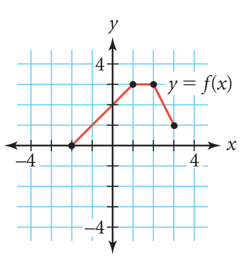
c. y is replaced with $(y - 2)$.

d. y is replaced with $(y + 2)$.

3. Each curve at right is a transformation of the graph of the parent function $y = \sqrt{x}$. Write an equation for each curve.



4. Given the graph of $y = f(x)$ below, draw a graph of each of these related functions.



a. $y = f(-x)$ **b.** $y = -f(x)$ **c.** $y = -f(-x)$

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Reason and Apply

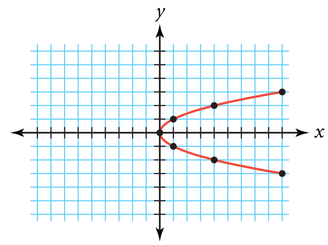
5. Consider the parent function $f(x) = \sqrt{x}$.

a. Name three pairs of integer coordinates that are on the graph of $y = f(x + 4) - 2$.

b. Write $y = f(x + 4) - 2$ using a **radical**, or square root symbol, and graph it.

c. Write $y = -f(x - 2) + 3$ using a radical, and graph it.

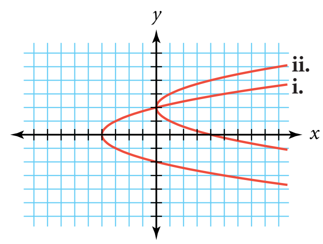
6. Consider the parabola at right:



a. Graph the parabola on your calculator. What two functions did you use?

b. Combine both functions from 6a using $\pm$ notation to create a single relation. Square both sides of the relation. What is the resulting equation?

7. Refer to the two parabolas shown.



a. Explain why neither graph represents a function.

b. Write a single equation for each parabola using $\pm$ notation.

c. Square both sides of each equation in 7b. What is the resulting equation of each parabola?

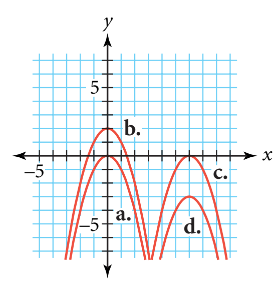
8. As Jake and Arthur travel together from Detroit to Chicago, each makes a graph relating time and distance. Jake, who lives in Detroit and keeps his watch on Detroit time, graphs his distance from Detroit. Arthur, who lives in Chicago and keeps his watch on Chicago time (1 hour earlier than Detroit), graphs his distance from Chicago. They both use the time shown on their watches for their x-axes. The distance between Detroit and Chicago is 250 miles.

a. Sketch what you think each graph might look like.

b. If Jake's graph is described by the function $y = f(x)$, what function describes Arthur's graph?

c. If Arthur's graph is described by the function $y = g(x)$, what function describes Jake's graph?

9. Write the equation of each parabola. Each parabola is a transformation of the graph of the parent function $y = x^2$.



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10. Write the equation of a parabola that is congruent to the graph of $y = -(x + 3)^2 + 4$, but translated right 5 units and down 2 units.

11. **APPLICATION** Police measure the lengths of skid marks to determine the initial speed of a vehicle before the brakes were applied. Many variables, such as the type of road surface and weather conditions, play an important role in determining the speed. The formula used to determine the initial speed is $S = 5.5\sqrt{D \cdot f}$ where S is the speed in miles per hour, D is the average length of the skid marks in feet, and f is a constant called the "drag factor." At a particular accident scene, assume it is known that the road surface has a drag factor of 0.7.

a. Write an equation that will determine the initial speed on this road as a function of the lengths of skid marks.

b. Sketch a graph of this function.

c. If the average length of the skid marks is 60 feet, estimate the initial speed of the car when the brakes were applied.

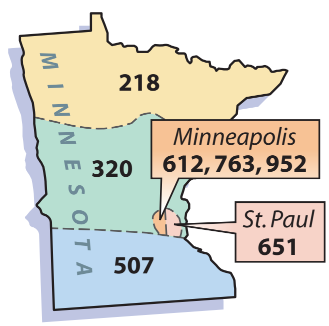
d. Solve your equation from 11a for D. What can you determine using this equation?

e. Graph your equation from 11d. What shape is it?

f. If you traveled on this road at a speed of 65 miles per hour and suddenly slammed on your brakes, how long would your skid marks be?

Review

12. Identify each relation that is also a function. For each relation that is not a function, explain why not.



a. independent variable: city
dependent variable: area code

b. independent variable: any pair of whole numbers
dependent variable: their greatest common factor

c. independent variable: any pair of fractions
dependent variable: their common denominator

d. independent variable: the day of the year
dependent variable: the time of sunrise

13. Solve for x. Solving square root equations often results in **extraneous solutions,** or answers that don't work in the original equation, so be sure to check your work.

a. $3 + \sqrt{x - 4} = 20$

b. $\sqrt{x + 7} = -3$

c. $4 - (x - 2)^2 = -21$

d. $5 - \sqrt{-(x + 4)} = 2$

14. Find the equation of the parabola with vertex $(-6, 4)$, a vertical line of symmetry, and containing the point $(-5, 5)$.

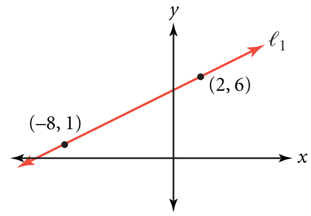
15. The graph of the line ℓ_1 is shown at right.

a. Write the equation of the line ℓ_1.

b. The line ℓ_2 is the image of the line ℓ_1 translated right 8 units. Sketch the line ℓ_2 and write its equation in a way that shows the horizontal translation.

c. The line ℓ_2 also can be thought of as the image of the line ℓ_1 after a vertical translation. Write the equation of the line ℓ_2 in a way that shows the vertical translation.

d. Show that the equations in 15b and 15c are equivalent.



16. Consider this data set:

{37, 40, 36, 37, 37, 49, 39, 47, 40, 38, 35, 46, 43, 40, 47, 49, 70, 65, 50, 73}

a. Give the five-number summary.

b. Display the data in a box plot.

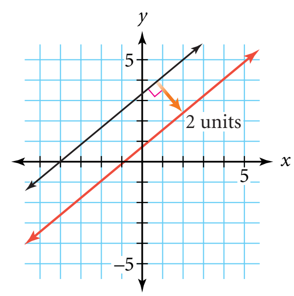
c. Find the interquartile range.

d. Identify any outliers, based on the interquartile range.

IMPROVING YOUR GEOMETRY SKILLS

Lines in Motion Revisited

Imagine that the graph of any line, $y = a + bx$, is translated 2 units in a direction perpendicular to it. What horizontal and vertical translations would be equivalent to this translation? What are the values of h and k? What is the linear equation of the image? You may want to use your calculator or geometry software to experiment with some specific linear equations before you try to generalize for h and k.



EXPLORATION

THE GEOMETER'S SKETCHPAD®

Rotation as a Composition of Transformations

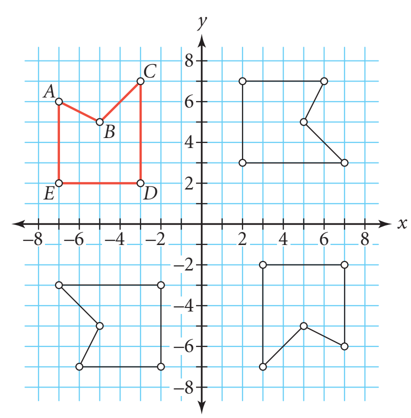
You have learned rules that reflect and translate figures and functions on the coordinate plane. Is it possible to rotate figures on a coordinate plane using a rule? You will explore that question in this activity.

Activity

Revolution

Step 1 Draw a figure using geometry software. Your figure should be nonsymmetric so that you can see the effects of various transformations.

Step 2 Rotate your figure three times: once by 90° counterclockwise, once by 90° clockwise, and once by 180° about the origin. Change your original figure to a different color.



Step 3 Transform your original figure onto each of the three images using only reflections and translations. (You may use other lines of reflection besides the axes.) Keep track of the transformations you use. Find at least two different sets of transformations that map the figure onto each of the three images.

Questions

1. Describe the effects of each rotation on the coordinates of the figure. Give a rule that describes the transformation of the x-coordinates and the y-coordinates for each of the three rotations. Do the rules change if your original figure is in a different quadrant?

2. Choose one of the combinations of transformations you found in Step 3. For each transformation you performed, explain the effect on the x- and y-coordinates. Show how the combination of these transformations confirms the rule you found by answering Question 1.

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LESSON

4.6

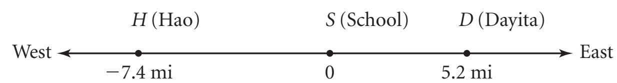
Keymath.com Links to Resources

Stretches and Shrinks and the Absolute-Value Family

A mind that is stretched by a new experience can never go back to its old dimensions.

OLIVER WENDELL HOLMES

Hao and Dayita ride the subway to school each day. They live on the same east-west subway route. Hao lives 7.4 miles west of the school, and Dayita lives 5.2 miles east of the school. This information is shown on the number line below.



The distance between two points is always positive. However, if you calculate Hao's distance from school, or *HS,* by subtracting his starting position from his ending position, you get a negative value:

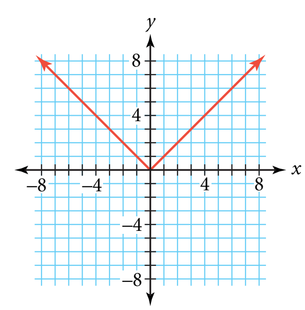
$$-7.4 - 0 = -7.4$$

In order to make the distance positive, you use the absolute-value function, which makes any input positive or zero. For example, the absolute value of -3 is 3, or $|-3| = 3$. For Hao's distance from school, you use the absolute-value function to calculate

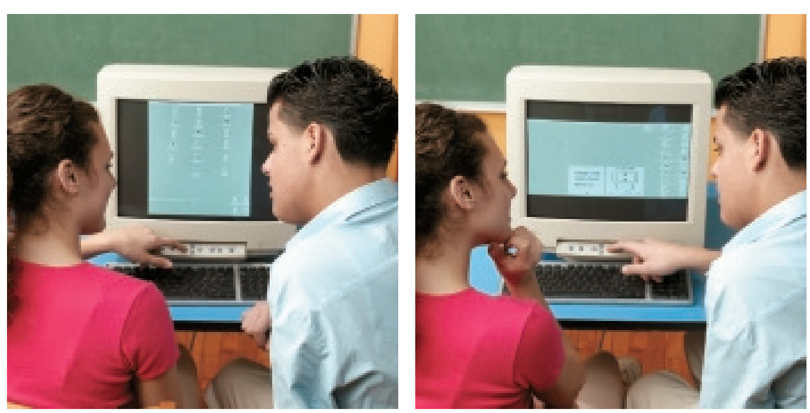
$$HS = |-7.4 - 0| = |-7.4| = 7.4$$

What is the distance from *D* to *H*? What is the distance from *H* to *D*?

In this lesson you will explore transformations of the graph of the parent function $y = |x|$. [▶ See **Calculator Note 4F** to learn how to graph the absolute-value function. ◀] You will write and use equations of the form $y = a\left|\frac{x-h}{b}\right| + k$. What you have learned about translating and reflecting other graphs will apply to these functions as well. You will also learn about transformations that **stretch** and **shrink** a graph.



Many computer and television screens have controls that allow you to change the scale of the horizontal or vertical dimension. Doing so stretches or shrinks the images on the screen.



EXAMPLE A

Graph the function $y = |x|$ with each of these functions. How does the graph of each function compare to the original graph?

a. $y = 2|x|$

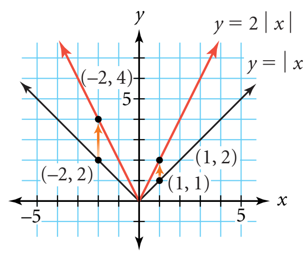
b. $y = \left|\frac{x}{3}\right|$

c. $y = 2\left|\frac{x}{3}\right|$

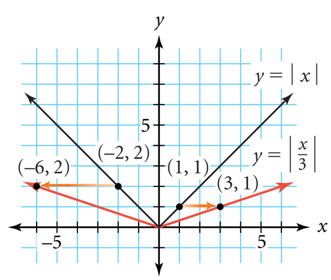
▶ Solution

In the graph of each function, the vertex remains at the origin. Notice, however, how the points (1, 1) and (−2, 2) on the parent function are mapped to a new location.

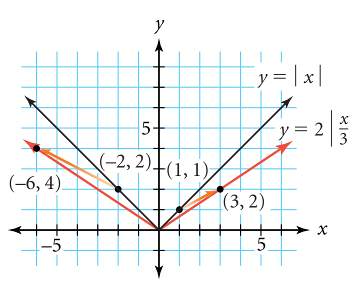
a. Every point on the graph of $y = 2|x|$ has a y-coordinate that is 2 times the y-coordinate of the corresponding point on the parent function. You say the graph of $y = 2|x|$ is a vertical stretch of the graph of $y = |x|$ by a factor of 2.



b. Replacing x with $\frac{x}{3}$ multiplies the x-coordinates by a factor of 3. The graph of $y = \left|\frac{x}{3}\right|$ is a horizontal stretch of the graph of $y = |x|$ by a factor of 3.



c. The combination of multiplying the parent function by 2 and dividing x by 3 results in a vertical stretch by a factor of 2 *and* a horizontal stretch by a factor of 3.



Translations and reflections are **rigid transformations**—they produce an image that is congruent to the original figure. Stretches and shrinks are **nonrigid transformations**—the image is not congruent to the original figure (unless you use a factor of 1 or -1). If you stretch or shrink a figure by the same **scale factor** both vertically and horizontally, then the image and the original figure will be similar, at least. If you stretch or shrink by different vertical and horizontal scale factors, then the image and the original figure will not be similar.

Robert Lazzarini, *payphone, 2002,* Mixed media, 108 x 84 x 48 in. (274.3 x 213.4 x 121.9 cm). As installed in 2002 Biennial Exhibition, Whitney Museum of Art, New York (March 7–May 26, 2002)

Using what you know about translations, reflections, and stretches, you can fit functions to data by locating only a few key points. For quadratic, square root, and absolute-value functions, first locate the vertex of the graph. Then use any other point to find the factors by which to stretch or shrink the image.

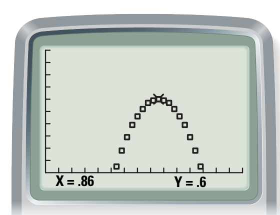
EXAMPLE B

These data are from one bounce of a ball. Find an equation that fits the data over this domain.

Time (s) x	Height (m) y
0.54	0.05
0.58	0.18
0.62	0.29
0.66	0.39
0.70	0.46
0.74	0.52
0.78	0.57
0.82	0.59
0.86	0.60

Time (s) x	Height (m) y
0.90	0.59
0.94	0.57
0.98	0.52
1.02	0.46
1.06	0.39
1.10	0.29
1.14	0.18
1.18	0.05

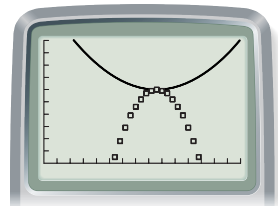
▶ Solution



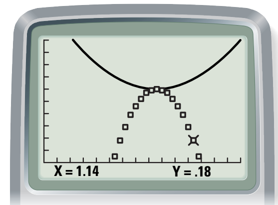
Graph the data on your calculator. The graph appears to be a parabola. However, the parent function $y = x^2$ has been reflected, translated, and possibly stretched or shrunken. Start by determining the translation. The vertex has been translated from (0, 0) to (0.86, 0.60). This is enough information for you to write the equation in the form $y = (x - h)^2 + k$, or $y = (x - 0.86)^2 + 0.60$. If you think of replacing x with $(x - 0.86)$ and replacing y with $(y - 0.60)$, you could also write the equivalent equation, $y - 0.6 = (x - 0.86)^2$.

The graph of $y = (x - 0.86)^2 + 0.60$ does not fit the data. The function still needs to be reflected and, as you can see from the graph, shrunken. Both of these transformations can be accomplished together.

Select one other data point to determine the scale factors, a and b. You can use any point, but you will get a better fit if you choose one that is not too close to the vertex. For example, you can choose the data point (1.14, 0.18).



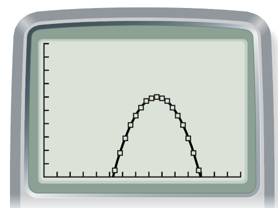
Assume this data point is the image of the point (1, 1) in the parent parabola $y = x^2$. In the graph of $y = x^2$, (1, 1) is 1 unit away from the vertex (0, 0) both horizontally and vertically. The data point we chose in this graph is $1.14 - 0.86$, or 0.28, unit away from the x-coordinate of the vertex, and $0.18 - 0.60$, or -0.42, unit away from the y-coordinate of the vertex. So, the horizontal scale factor is 0.28, and the vertical scale factor is -0.42. The negative vertical scale factor also produces a reflection across the x-axis.



Combine these scale factors with the translations to get the final equation

$$\frac{y - 0.6}{-0.42} = \left(\frac{x - 0.86}{0.28}\right)^2 \quad \text{or} \quad y = -0.42\left(\frac{x - 0.86}{0.28}\right)^2 + 0.6$$

This model, shown at right, fits the data nicely.



The same procedure works with the other functions you have studied so far. As you continue to add new functions to your mathematical knowledge, you will find that what you have learned about function transformations continues to apply.

Investigation
The Pendulum

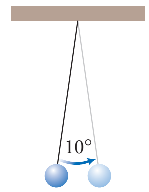
You will need

- string
- a small weight
- a stopwatch or a watch with a second hand

Italian mathematician and astronomer Galileo Galilei (1564–1642) made many contributions to our understanding of gravity, the physics of falling objects, and the orbits of the planets. One of his famous experiments involved the periodic motion of a pendulum. In this investigation you will carry out the same experiment and find a function to model the data.

This fresco, painted in 1841, shows Galileo at age 17, contemplating the motion of a swinging lamp in the Cathedral of Pisa. A swinging lamp is an example of a pendulum.

Step 1 Follow the Procedure Note to find the period of your pendulum. Repeat the experiment for several different string lengths and complete a table of values. Use a variety of short, medium, and long string lengths.



Step 2 Graph the data using *length* as the independent variable. What is the shape of the graph? What do you suppose is the parent function?

Step 3 The vertex is at the origin, (0, 0). Why do you suppose it is there?

Step 4 Divide up your data points and have each person in your group find the horizontal or vertical stretch or shrink from the parent function. Apply these transformations to find an equation to fit the data.

Step 5 Compare the collection of equations from your group. Which points are the best to use to fit the curve? Why do these points work better than others?

Procedure Note

1. Tie a weight at one end of a length of string to make a pendulum. Firmly hold the other end of the string, or tie it to something, so that the weight hangs freely.
2. Measure the length of the pendulum, from the center of the weight to the point where the string is held.
3. Pull the weight to one side and release it so that it swings back and forth in a short arc, about 10° to 20°. Time ten complete swings (forward and back is one swing).
4. The **period** of your pendulum is the time for one complete swing (forward and back). Find the period by dividing by 10.

In the exercises you will use techniques you discovered in this lesson. Remember that replacing y with $\frac{y}{a}$ stretches a graph by a factor of a vertically. Replacing x with $\frac{x}{b}$ stretches a graph by a factor of b horizontally. When graphing a function, you should do stretches and shrinks before translations to avoid moving the vertex.

Stretch or Shrink of a Function

A **stretch** or a **shrink** is a transformation that expands or compresses a graph either horizontally or vertically.

Given the graph of $y = f(x)$, the graph of

$$\frac{y}{a} = f(x) \quad \text{or} \quad y = af(x)$$

is a vertical stretch or shrink by a factor of a. When $a > 1$, it is a stretch; when $0 < a < 1$, it is a shrink. When $a < 0$, a reflection across the x-axis also occurs.

Given the graph of $y = f(x)$, the graph of

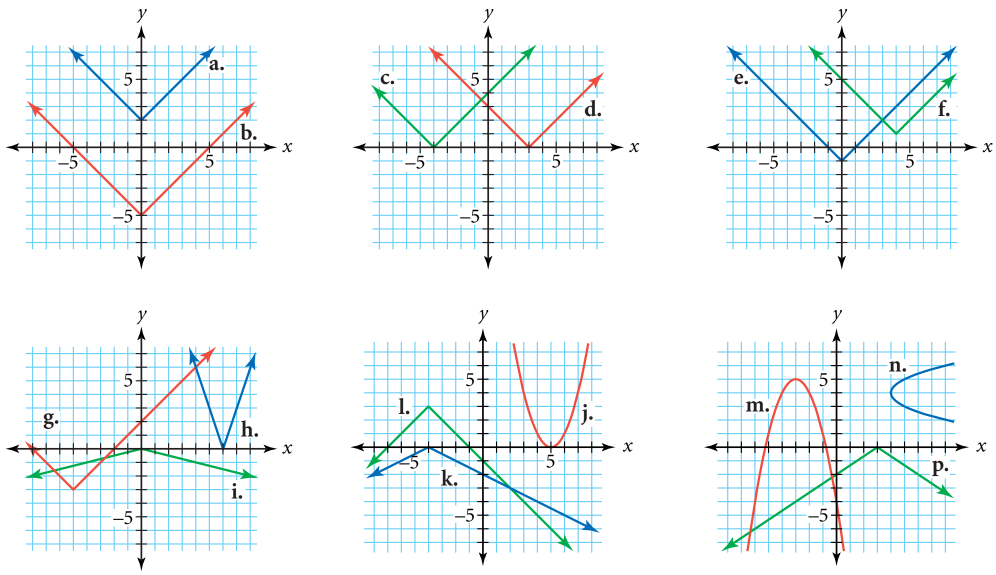
$$y = f\left(\frac{x}{b}\right) \quad \text{or} \quad y = f\left(\frac{1}{b} \cdot x\right)$$

is a horizontal stretch or shrink by a factor of b. When $b > 1$, it is a stretch; when $0 < b < 1$, it is a shrink. When $b < 0$, a reflection across the y-axis also occurs.

EXERCISES

Practice Your Skills

1. Each graph is a transformation of the graph of one of the parent functions you've studied. Write an equation for each graph.



2. Describe what happens to the graph of $y = f(x)$ in these situations.

a. x is replaced with $\frac{x}{3}$. **b.** x is replaced with $-x$. **c.** x is replaced with $3x$.

d. y is replaced with $\frac{y}{2}$. **e.** y is replaced with $-y$. **f.** y is replaced with $2y$.

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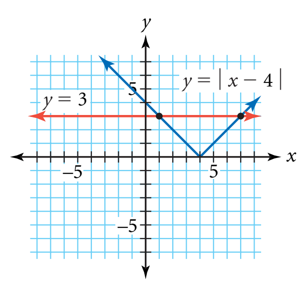
3. Solve each equation for y.

a. $y + 3 = 2(x - 5)^2$ **b.** $\frac{y+5}{2} = \left|\frac{x+1}{3}\right|$ **c.** $\frac{y+7}{-2} = \sqrt{\frac{x-6}{-3}}$

Reason and Apply

4. Choose a few different values for a. What can you conclude about $y = a|x|$ and $y = |ax|$? Are they the same function?

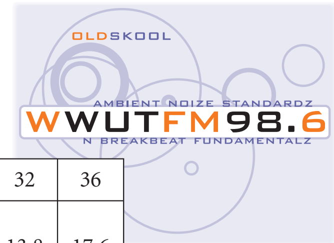
5. The graph at right shows how to solve the equation $|x - 4| = 3$ graphically. The equations $y = |x - 4|$ and $y = 3$ are graphed on the same coordinate axes.



a. What is the x-coordinate of each point of intersection? What x-values are solutions of the equation $|x - 4| = 3$?

b. Solve the equation $|x + 3| = 5$ graphically.

6. **APPLICATION** You can use a single radio receiver to find the distance to a transmitter by measuring the strength of the signal. Suppose these approximate distances are measured with a receiver while you drive along a straight road. Find a model that fits the data. Where do you think the transmitter might be located?



Miles traveled	0	4	8	12	16	20	24	28	32	36
Distance from transmitter (miles)	18.4	14.4	10.5	6.6	2.5	1.8	6.0	9.9	13.8	17.6

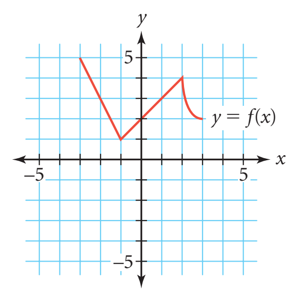
7. Assume that you know the vertex of a parabola is $(5, -4)$.
   a. If the parabola is stretched vertically by a factor of 2 in relation to the graph of $y = x^2$, what are the coordinates of the point 1 unit to the right of the vertex?
   b. If the parabola is stretched horizontally by a factor of 3 in relation to the graph of $y = x^2$, what are the coordinates of the points 1 unit above the vertex?
   c. If the parabola is stretched vertically by a factor of 2 and horizontally by a factor of 3, name two points that are symmetric with respect to the vertex.

8. Given the parent function $y = x^2$, describe the transformations represented by the function $\frac{y-2}{3} = \left(\frac{x+7}{4}\right)^2$. Sketch a graph of the transformed parabola.

9. A parabola has vertex $(4.7, 5)$ and passes through the point $(2.8, 9)$.
   a. What is the equation of the axis of symmetry for this parabola?
   b. What is the equation of this parabola?
   c. Is this the only parabola passing through this vertex and point? Explain. Sketch a graph to support your answer.

10. Sketch a graph of each of these equations.
   a. $\frac{y-2}{3} = (x-1)^2$
   b. $\left(\frac{y+1}{2}\right)^2 = \frac{x-2}{3}$
   c. $\frac{y-2}{2} = \left|\frac{x+1}{3}\right|$

11. Given the graph of $y = f(x)$, draw graphs of these related functions.
   a. $\frac{y}{-2} = f(x)$
   b. $y = f\left(\frac{x-3}{2}\right)$
   c. $\frac{y+1}{\frac{1}{2}} = f(x+1)$



12. **APPLICATION** A chemistry class gathered these data on the conductivity of a base solution as acid is added to it. Graph the data and use transformations to find a model to fit the data.

Acid volume (mL) x	Conductivity (μS/cm^3) y
0	4152.95
1	3140.97
2	2100.34
3	1126.55
4	162.299

Acid volume (mL) x	Conductivity (μS/cm^3) y
5	1212.47
6	2358.11
7	3417.83
8	4429.81

Review

13. A panel of judges rate 20 science fair exhibits as shown. The judges decide that the top rating should be 100, so they add 6 points to each rating.

a. What are the mean and the standard deviation of the ratings before adding 6 points?

b. What are the mean and the standard deviation of the ratings after adding 6 points?

c. What do you notice about the change in the mean? In the standard deviation?

Exhibit number	Rating	Exhibit number	Rating
1	79	11	85
2	81	12	88
3	94	13	86
4	92	14	83
5	68	15	89
6	79	16	90
7	71	17	92
8	83	18	77
9	89	19	84
10	92	20	73

14. **APPLICATION** This table shows the percentage of households with computers in the United States in various years.

Year	1995	1996	1997	1998	1999	2000
Households (%)	31.7	35.5	39.2	42.6	48.2	53.0

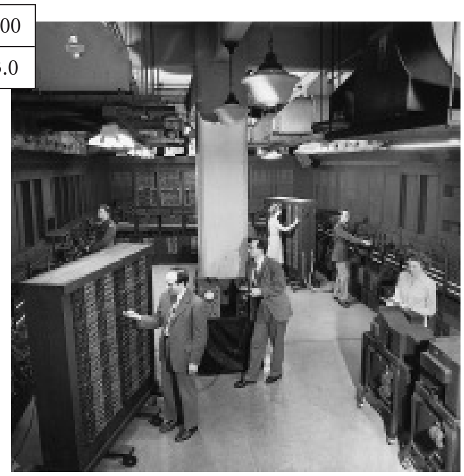
(*The New York Times Almanac 2002*)

a. Make a scatter plot of these data.

b. Find the median-median line.

c. Use the median-median line to predict the percentage of households with computers in 2002.

d. Is a linear model a good model for this situation? Explain your reasoning.



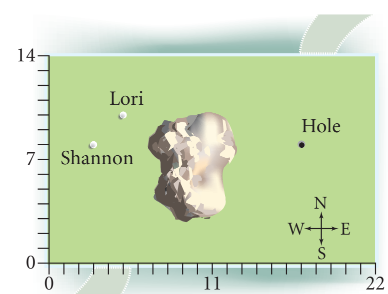
In 1946, inventors J. Presper Eckert and J. W. Mauchly created the first general-purpose electronic calculator, named ENIAC (Electronic Numerical Integrator and Computer). The calculator filled a large room and required a team of engineers and maintenance technicians to operate it.

IMPROVING YOUR VISUAL THINKING SKILLS

Miniature Golf

Shannon and Lori are playing miniature golf. The third hole is in a 22-by-14-foot walled rectangular playing area with a large rock in the center. Each player's ball comes to rest as shown. The rock makes a direct shot into the hole impossible.

At what point on the south wall should Shannon aim in order to have the ball bounce off and head directly for the hole? Recall from geometry that the angle of incidence is equal to the angle of reflection. Lori cannot aim at the south wall. Where should she aim?



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LESSON

4.7

Keymath.com Links to Resources

Transformations and the Circle Family

Many times the best way, in fact the only way, to learn is through mistakes. A fear of making mistakes can bring individuals to a standstill, to a dead center.

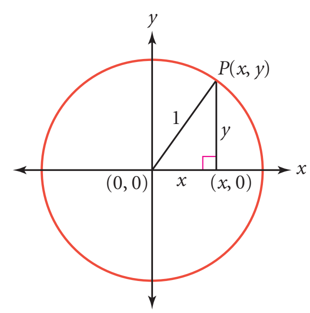
GEORGE BROWN

You have explored several functions and relations and transformed them in a plane. You know that a horizontal translation occurs when x is replaced with $(x - h)$ and that a vertical translation occurs when y is replaced with $(y - k)$. You have reflected graphs across the y-axis by replacing x with $-x$ and across the x-axis by replacing y with $-y$. You have also stretched and shrunk a function vertically by replacing y with $\frac{y}{a}$, and horizontally by replacing x with $\frac{x}{b}$.

This photo shows circular housing developments in Denmark.

In this lesson you will stretch and shrink the graph of a relation that is not a function and discover how to create the equation for a new shape.

You will start by investigating the circle. A **unit circle** has a radius of 1 unit. Suppose P is any point on a unit circle with center at the origin. Draw the slope triangle between the origin and point P.



You can derive the equation of a circle from this diagram by using the Pythagorean Theorem. The legs of the right triangle have lengths x and y and the length of the hypotenuse is 1 unit, so its equation is $x^2 + y^2 = 1$. This is true for all points P on the unit circle.

Equation of a Unit Circle

The equation of a **unit circle** with center (0, 0) is

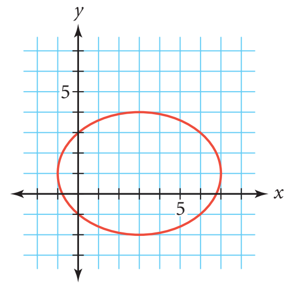
$$x^2 + y^2 = 1$$

What are the domain and the range of this circle? If a value, such as 0.5, is substituted for x, what are the output values of y? Is this the graph of a function? Why or why not?

In order to draw the graph of a circle on your calculator, you need to solve the equation $x^2 + y^2 = 1$ for y. When you do this, you get two equations, $y = +\sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$. Each of these is a function. You have to graph both of them to get the complete circle.

You can transform a circle to get an **ellipse.** An ellipse is a stretched or shrunken circle.

EXAMPLE A What is the equation of this ellipse?



▶ Solution

The original unit circle has been translated and stretched both horizontally and vertically. The new center is at (3, 1). In a unit circle, every radius measures 1 unit. In this ellipse, a horizontal segment from the center to the ellipse measures 4 units, so the horizontal scale factor is 4. Likewise, a vertical segment from the center to the ellipse measures 3 units, so the vertical scale factor is 3. So the equation changes like this:

$$x^2 + y^2 = 1$$

Original unit circle.

$$\left(\frac{x}{4}\right)^2 + y^2 = 1$$

Stretch horizontally by a factor of 4. (Replace x with $\frac{x}{4}$.)

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Stretch vertically by a factor of 3. (Replace y with $\frac{y}{3}$.)

$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$$

Translate to new center at (3, 1). (Replace x with $x - 3$, and replace y with $y - 1$.)

To enter this equation into your calculator to check your answer, you need to solve for y.

$$\left(\frac{y-1}{3}\right)^2 = 1 - \left(\frac{x-3}{4}\right)^2$$

Subtract $\left(\frac{x-3}{4}\right)^2$ from both sides.

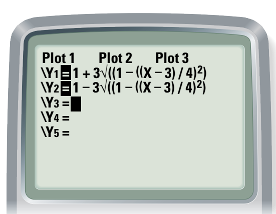
$$\frac{y-1}{3} = \pm\sqrt{1 - \left(\frac{x-3}{4}\right)^2}$$

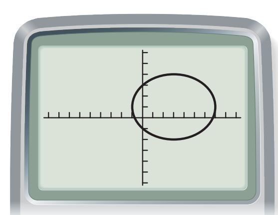
Take the square root of both sides.

$$y = 1 \pm 3\sqrt{1 - \left(\frac{x-3}{4}\right)^2}$$

Multiply both sides by 3, then add 1.

It takes two equations to graph this on your calculator. By graphing both of these equations, you can draw the complete ellipse and verify your answer.





[−9.4, 9.4, 1, −6.2, 6.2, 1]

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Investigation
When Is a Circle Not a Circle?

You will need

- the worksheet When Is a Circle Not a Circle?

If you look at a circle, like the top rim of a cup, from an angle, you don't see a circle; you see an ellipse. Choose one of the ellipses from the worksheet. Use your ruler carefully to place axes on the ellipse, and scale your axes in centimeters. Be sure to place the axes so that the longest dimension is parallel to one of the axes. Find the equation to model your ellipse. Graph your equation on your calculator and verify that it creates an ellipse with the same dimensions as on the worksheet.

The tops of these circular oil storage tanks look elliptical when viewed at an angle.

Equations for transformations of relations such as circles and ellipses are sometimes easier to work with in the general form before you solve them for y, but you need to solve for y to enter the equations into your calculator. If you start with a function such as the top half of the unit circle, $f(x) = \sqrt{1 - x^2}$, you can transform it in the same way you transformed any other function, but it may be a little messier to deal with.

EXAMPLE B

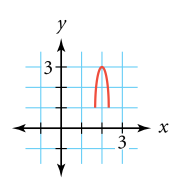
If $f(x) = \sqrt{1 - x^2}$, find $g(x) = 2f(3(x - 2)) + 1$. Sketch a graph of this new function.

▶ Solution

In $g(x) = 2f(3(x - 2)) + 1$, note that $f(x)$ is the parent function, x has been replaced with $3(x - 2)$, and $f(3(x - 2))$ is then multiplied by 2 and 1 is added. You can rewrite the function g as

$$g(x) = 2\sqrt{1 - (3(x - 2))^2} + 1 \quad \text{or} \quad g(x) = 2\sqrt{1 - \left(\frac{x - 2}{\frac{1}{3}}\right)^2} + 1$$

This indicates that the graph of $y = f(x)$, a semicircle, has been shrunken horizontally by a factor of $\frac{1}{3}$, stretched vertically by a factor of 2, then translated right 2 units and up 1 unit. The transformed semicircle is graphed at right. What are the coordinates of the right endpoint of the graph? Describe how the original semicircle's right endpoint of (1, 0) was mapped to this new location.



You have now learned to translate, reflect, stretch, and shrink functions and relations. These transformations are the same for all equations.

Transformations of Functions and Relations

Translations

The graph of $y = k + f(x - h)$ translates the graph of $y = f(x)$ h units horizontally and k units vertically.

or

Replacing x with $(x - h)$ translates the graph h units horizontally.
Replacing y with $(y - k)$ translates the graph k units vertically.

Reflections

The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ across the y-axis. The graph of $y = -f(x)$ is a reflection the graph of $y = f(x)$ across the x-axis.

or

Replacing x with $-x$ reflects the graph across the y-axis. Replacing y with $-y$ reflects the graph across the x-axis.

Stretches and Shrinks

The graph of $y = af\left(\frac{x}{b}\right)$ is a stretch or shrink of the graph of $y = f(x)$ by a vertical scale factor of a and by a horizontal scale factor of b.

or

Replacing x with $\frac{x}{b}$ stretches or shrinks the graph by a horizontal scale factor of b. Replacing y with $\frac{y}{a}$ stretches or shrinks the graph by a vertical scale factor of a.

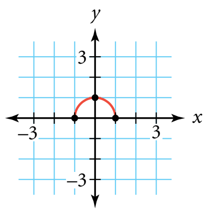
EXERCISES

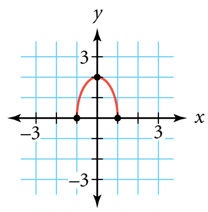
Practice Your Skills

1. Each equation represents a single transformation. Copy and complete this table.

Equation	Transformation (translation, reflection, stretch, shrink)	Direction	Amount or scale factor
$y + 3 = x^2$	Translation	Down	3
$-y = \lvert x \rvert$			
$y = \sqrt{\frac{x}{4}}$			
$\frac{y}{0.4} = x^2$			
$y = \lvert x - 2 \rvert$			
$y = \sqrt{-x}$			

2. The equation $y = \sqrt{1 - x^2}$ is the equation of the top half of the unit circle with center (0, 0) shown on the left. What is the equation of the top half of an ellipse shown on the right?



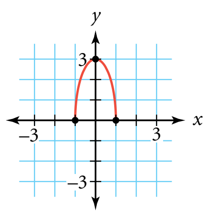


3. Use $f(x) = \sqrt{1 - x^2}$ to graph each of the transformations below.

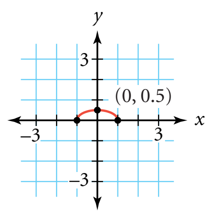
a. $g(x) = -f(x)$ **b.** $h(x) = -2f(x)$ **c.** $j(x) = -3 + 2f(x)$

4. Each curve is a transformation of the graph of $y = \sqrt{1 - x^2}$. Write an equation for each curve.

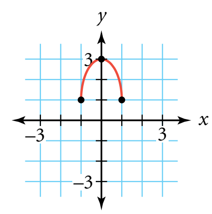
a.



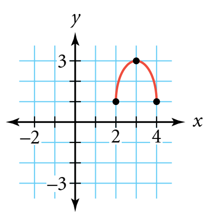
b.



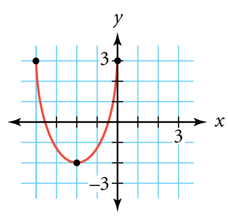
c.



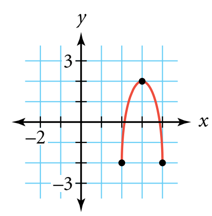
d.



e.



f.



5. Write an equation and draw a graph for each transformation of the unit circle. Use the form $y = \pm\sqrt{1 - x^2}$.

Keymath.com More Practice Your Skills

a. Replace y with $(y - 2)$.

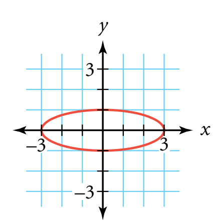
b. Replace x with $(x + 3)$.

c. Replace y with $\frac{y}{2}$.

d. Replace x with $\frac{x}{2}$.

Reason and Apply

6. To create the ellipse at right, the x-coordinate of each point on a unit circle has been multiplied by a factor of 3.

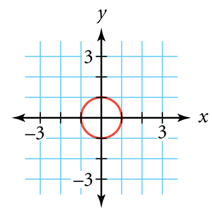


a. Write the equation of this ellipse.

b. What expression did you substitute for x in the parent equation?

c. If $y = f(x)$ is the function for the top half of a unit circle, then what is the function for the top half of this ellipse, $y = g(x)$, in terms of f?

7. Given the unit circle at right, write the equation that generates each transformation. Use the form $x^2 + y^2 = 1$.

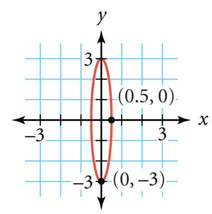


a. Each y-value is half the original y-value.

b. Each x-value is half the original x-value.

c. Each y-value is half the original y-value, and each x-value is twice the original x-value.

8. Consider the ellipse at right.



a. Write two functions that you could use to graph this ellipse.

b. Use ± to write one equation that combines the two equations in 8a.

c. Write another equation for the ellipse by squaring both sides of the equation in 8b.

9. ***Mini-Investigation*** Follow these steps to explore a relationship between linear, quadratic, square root, absolute-value, and semicircle functions. Use friendly windows of an appropriate size.

a. Graph these equations simultaneously on your calculator. The first four functions intersect in the same two points. What are the coordinates of these points?

$y = x \qquad y = x^2 \qquad y = \sqrt{x} \qquad y = |x| \qquad y = \sqrt{1 - x^2}$

b. Imagine using the intersection points that you found in 9a to draw a rectangle that just encloses the quarter-circle that is on the right half of the fifth function. How do the coordinates of the points relate to the dimensions of the rectangle?

c. Solve these equations for y and graph them simultaneously on your calculator. Where do the first four functions intersect?

$\frac{y}{2} = \frac{x}{4} \qquad \frac{y}{2} = \left(\frac{x}{4}\right)^2 \qquad \frac{y}{2} = \sqrt{\frac{x}{4}} \qquad \frac{y}{2} = \left|\frac{x}{4}\right| \qquad \frac{y}{2} = \sqrt{1 - \left(\frac{x}{4}\right)^2}$

d. Imagine using the intersection points that you found in 9c to draw a rectangle that just encloses the right half of the fifth function. How do the coordinates of the points relate to the dimensions of the rectangle?

e. Solve these equations for y and graph them simultaneously on your calculator. Where do the first four functions intersect?

i. $\frac{y-3}{2} = \frac{x-1}{4}$ ii. $\frac{y-3}{2} = \left(\frac{x-1}{4}\right)^2$ iii. $\frac{y-3}{2} = \sqrt{\frac{x-1}{4}}$

iv. $\frac{y-3}{2} = \left|\frac{x-1}{4}\right|$ v. $\frac{y-3}{2} = \sqrt{1 - \left(\frac{x-1}{4}\right)^2}$

f. What are the dimensions of a rectangle that encloses the right half of the fifth function? How do these dimensions relate to the coordinates of the two points in 9e?

g. In each set of functions, one of the points of intersection located the center of the transformed semicircle. How did the other points relate to the shape of the semicircle?

Science
CONNECTION

Satellites are used to aid in navigation, communication, research, and military reconnaissance. The job the satellite is meant to do will determine the type of orbit it is placed in.

A satellite in geosynchronous orbit moves in an east-west direction and always stays directly over the same spot on Earth, so its orbital path is circular. The satellite and Earth move together, so both orbits take 24 hours. Because we always know where the satellite is, satellite dish antennae on Earth can be aimed in the right direction.

Another useful orbit is a north-south elliptical orbit that takes 12 hours to circle the planet. Satellites in these elliptical orbits cover areas of Earth that are not covered by geosynchronous satellites, and are therefore more useful for research and reconnaissance.

Satellites in a geosynchronus orbit follow a circular path above the equator. Another common orbit is an elliptical orbit in the north-south direction. For more information, see the links at www.keymath.com/DAA .

Review

10. Refer to Exercise 13 in Lesson 4.6. The original data is shown at right. Instead of adding the same number to each score, one of the judges suggests that perhaps they should *multiply* the original scores by a factor that makes the highest score equal 100. They decide to try this method.

a. By what factor should they multiply the highest score, 94, to get 100?

b. What are the mean and the standard deviation of the original ratings? Of the altered ratings?

c. Let x represent the exhibit number, and let y represent the rating. Plot the original and altered ratings on the same graph. Describe what happened to the ratings visually. How does this explain what happened to the mean and the standard deviation?

d. Which method do you think the judges should use? Explain your reasoning.

Exhibit number	Rating
1	79
2	81
3	94
4	92
5	68
6	79
7	71
8	83
9	89
10	92

Exhibit number	Rating
11	85
12	88
13	86
14	83
15	89
16	90
17	92
18	77
19	84
20	73

11. Find the next three terms in this sequence: 16, 40, 100, 250, . . .

12. Solve. Give answers to the nearest 0.01.

a. $\sqrt{1-(a-3)^2} = 0.5$

b. $-4\sqrt{1-(b+2)^2} = -1$

c. $\sqrt{1-\left(\frac{c-2}{3}\right)^2} = 0.8$

d. $3 + 5\sqrt{1-\left(\frac{d+1}{2}\right)^2} = 8$

13. This table shows the distances needed to stop a car on dry pavement in a minimum length of time for various speeds. Reaction time is assumed to be 0.75 s.

Speed (mi/h) x	10	20	30	40	50	60	70
Stopping distance (ft) y	19	42	73	116	173	248	343

a. Construct a scatter plot of these data.

b. Find the equation of a parabola that fits the points and graph it.

c. Find the residuals for this equation and the root mean square error.

d. Predict the stopping distance for 56.5 mi/h.

e. How close should your prediction in 13d be to the *actual* stopping distance?

14. This table shows passenger activity in the world's 30 busiest airports in 2000.

a. Display the data in a histogram.

b. Estimate the total number of passengers who used the 30 airports. Explain any assumptions you make.

c. Estimate the mean usage among the 30 airports in 2000. Mark the mean on your histogram.

d. Sketch a box plot above your histogram. Estimate the five-number summary values. Explain any assumptions you make.

Number of passengers (in millions)	Number of airports
$25 \leq p < 30$	5
$30 \leq p < 35$	8
$35 \leq p < 40$	8
$40 \leq p < 45$	1
$45 \leq p < 50$	2
$55 \leq p < 60$	1
$60 \leq p < 65$	2
$65 \leq p < 70$	1
$70 \leq p < 75$	1
$80 \leq p < 85$	1

(*The New York Times Almanac 2002*)

15. Consider the linear function $y = 3x + 1$.

a. Write the equation of the image of the graph of $y = 3x + 1$ after a reflection across the x-axis. Graph both lines on the same axes.

b. Write the equation of the image of the graph of $y = 3x + 1$ after a reflection across the y-axis. Graph both lines on the same axes.

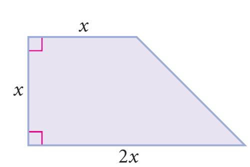
c. Write the equation of the image of the graph of $y = 3x + 1$ after a reflection across the x-axis and then across the y-axis. Graph both lines on the same axes.

d. How does the image in 14c compare to the original line?

IMPROVING YOUR VISUAL THINKING SKILLS

4-in-1

Copy this trapezoid. Divide it into four congruent polygons.



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LESSON 4.8

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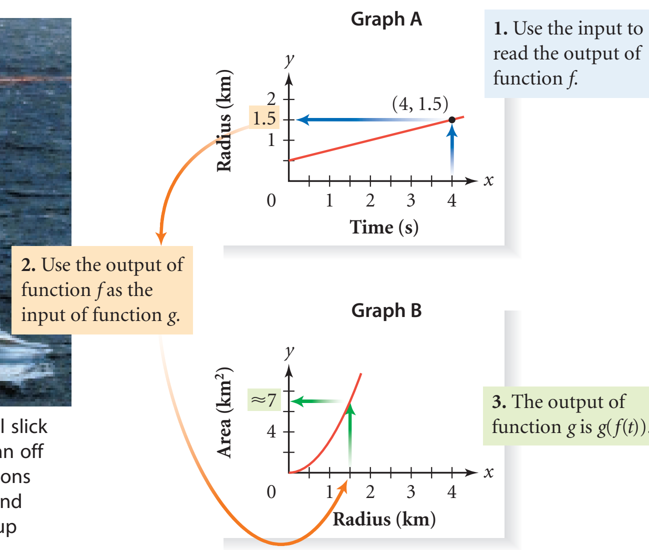
Compositions of Functions

Sometimes you'll need two or more functions in order to answer a question or analyze a problem. Suppose an offshore oil well is leaking. Graph A shows the radius, r, of the spreading oil slick, growing as a function of time, t, so $r = f(t)$. Graph B shows the area, a, of the circular oil slick as a function of its radius, r, so $a = g(r)$. Time is measured in hours, the radius is measured in kilometers, and the area is measured in square kilometers.

This French Navy ship is attempting to surround an oil slick after the *Erika* oil tanker broke up in the Atlantic Ocean off the western coast of France in 1999. Three million gallons of oil poured into the ocean, killing 16,000 sea birds and polluting 250 miles of coastline. The cost of the cleanup efforts exceeded $160 million.



If you want to find the area of the oil slick after 4 hours, you use function f on Graph A to find that when t equals 4, r equals 1.5. Next, using function g on Graph B, you find that when r equals 1.5, a is approximately 7. So after 4 h, the radius of the oil slick is 1.5 km and its area is 7 km^2.

You used the graphs of two different functions, f and g, to find that after 4 h, the oil slick has area 7 km^2. You actually used the output from one function, f, as the input in the other function, g. This is an example of a **composition of functions** to form a new functional relationship between area and time, that is, $a = g(f(t))$. The symbol $g(f(t))$, read "g of f of t," is a composition of the two functions f and g. The composition $g(f(t))$ gives the final outcome when an x-value is substituted into the "inner" function, f, and its output value, $f(t)$, is then substituted as the input into the "outer" function, g.

EXAMPLE A

Consider these functions:

$$f(x) = \frac{3}{4}x - 3 \quad \text{and} \quad g(x) = |x|$$

What will the graph of $y = g(f(x))$ look like?

▶ Solution

Function f is the inner function, and function g is the outer function. Use equations and tables to identify the output of f and use it as the input of g.

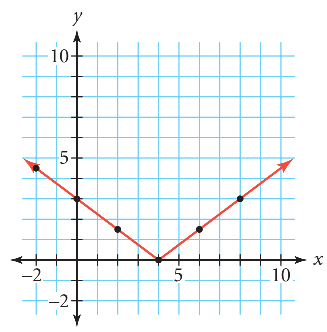
Find several $f(x)$ output values.	Use the $f(x)$ output values as the input of $g(x)$.	Match the input of the inner function, f, with the output of the outer function, g, and plot the graph.

x	$f(x)$
-2	-4.5
0	-3
2	-1.5
4	0
6	1.5
8	3

$f(x)$	$g(f(x))$
-4.5	4.5
-3	3
-1.5	1.5
0	0
1.5	1.5
3	3

x	$g(f(x))$
-2	4.5
0	3
2	1.5
4	0
6	1.5
8	3

The solution is the composition graph at right. All the function values of f, whether positive or negative, give positive output values under the rule of g, the absolute-value function. So, the part of the graph of function f showing negative output values is reflected across the x-axis in this composition.



You can use what you know about transformations to get the specific equation for $y = g(f(x))$ in Example A. Use the parent function $y = |x|$, translate the vertex right 4 units, and then stretch horizontally by a factor of 4 and vertically by a factor of 3. This gives the equation $y = 3\left|\frac{x-4}{4}\right|$. You can algebraically manipulate this equation to get $y = \left|\frac{3}{4}x - 3\right|$, which appears to be the equation of f substituted for the input of g. You can always create equations of composed functions by substituting one equation into another.

Investigation
Looking Up

You will need

- a small mirror
- one or more tape measures or metersticks

First, you'll establish a relationship between your distance from a mirror and what you can see in it.

Procedure Note

1. Place the mirror flat on the floor 0.5 m from a wall.
2. Use tape to attach tape measures or metersticks up the wall to a height of 1.5 to 2 m.

Step 1 Set up the experiment as in the Procedure Note. Stand a short distance from the mirror, and look down into it. Move slightly left or right until you can see the tape measure on the wall reflected in the mirror.

Step 2 Have a group member slide his or her finger up the wall to help locate the highest height mark that is reflected in the mirror. Record the height in centimeters, h, and the distance from your toe to the center of the mirror in centimeters, d.

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Step 3 | Change your distance from the mirror and repeat Step 2. Make sure you keep your head in the same position. Collect several pairs of data in the form (d, h). Include some distances from the mirror that are short and some that are long.

Step 4 | Find a function that fits your data by transforming the parent function $h = \frac{1}{d}$. Call this function f.

Now you'll combine your work from Steps 1–4 with the scenario of a timed walk toward and away from the mirror.

Step 5 | Suppose this table gives your position at 1-second intervals:

Time (s) t	0	1	2	3	4	5	6	7
Distance to mirror (cm) d	163	112	74	47	33	31	40	62

Use one of the families of functions from this chapter to fit these data. Call this function g. It should give the distance from the mirror for seconds 0 to 7.

Step 6 | Use your two functions to answer these questions:

a. How high up the wall can you see when you are 47 cm from the mirror?

b. Where are you at 1.3 seconds?

c. How high up the wall can you see at 3.4 seconds?

Step 7 | Change each expression into words relating to the context of this investigation and find an answer. Show the steps you needed to evaluate each expression.

a. $f(60)$

b. $g(5.1)$

c. $f(g(2.8))$

Step 8 | Find a single function, $H(t)$, that does the work of $f(g(t))$. Show that $H(2.8)$ gives the same answer as Step 7c above.

Don't confuse a composition of functions with the product of functions. Composing functions requires you to replace the independent variable in one function with the output value of the other function. This means that it is generally not commutative. That is, $f(g(x)) \neq g(f(x))$, except for certain functions.

You can compose a function with itself. The next example shows you how.

EXAMPLE B

Suppose the function $A(x) = \left(1 + \frac{0.07}{12}\right)x - 250$ gives the balance of a loan with an annual interest rate of 7%, compounded monthly, in the month after a \$250 payment. In the equation, x represents the current balance and $A(x)$ represents the next balance. Translate these expressions into words and find their values.

a. $A(15000)$

b. $A(A(20000))$

c. $A(A(A(18000)))$

d. $A(A(x))$

▶ Solution

Each expression builds from the inside out.

a. $A(15000)$ asks, "What is the loan balance after one monthly payment if the starting balance is \$15,000?" Substituting 15000 for x in the given equation, you get $A(15000) = \left(1 + \frac{0.07}{12}\right)15000 - 250 = 14837.50$, or \$14,837.50.

b. $A(A(20000))$ asks, "What is the loan balance after two monthly payments if the starting balance is \$20,000?" Substitute 20000 for x in the given equation. You get 19866.67 and use it as input in the given equation. That is, $A(A(20000)) = A(19866.67) = 19732.56$, or \$19,732.56.

c. $A(A(A(18000)))$ asks, "What is the loan balance after three monthly payments if the starting balance is \$18,000?" Working from the inner expression outward, you get $A(A(A(18000)) = A(A(17855)) = A(17709.15) = 17562.46$, or \$17,562.46.

d. $A(A(x))$ asks, "What is the loan balance after two monthly payments if the starting balance is x?"

$$A(A(x)) = A\left(\left(1 + \frac{0.07}{12}\right)x - 250\right)$$

Use the given function to substitute $\left(1 + \frac{0.07}{12}\right)x - 250$ for $A(x)$.

$$= \left(1 + \frac{0.07}{12}\right)\left[\left(1 + \frac{0.07}{12}\right)x - 250\right] - 250$$

Use the output, $\left[\left(1 + \frac{0.07}{12}\right)x - 250\right]$ in place of the input, x.

$$= 1.005833(1.005833x - 250) - 250$$

Convert the fractions to decimal approximations.

$$= 1.0117x - 251.458 - 250$$

Apply the distributive property.

$$= 1.0117x - 501.458$$

Subtract.

EXERCISES

Practice Your Skills

1. Given the functions $f(x) = 3 + \sqrt{x + 5}$ and $g(x) = 2 + (x - 1)^2$, find these values.

a. $f(4)$ **b.** $f(g(4))$ **c.** $g(-1)$ **d.** $g(f(-1))$

2. The functions f and g are defined by these sets of input and output values.

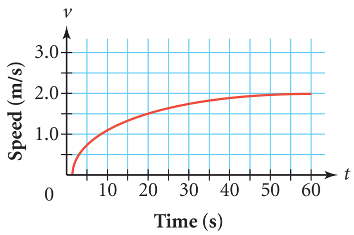
$g = \{(1, 2), (-2, 4), (5, 5), (6, -2)\}$

$f = \{(0, -2), (4, 1), (3, 5), (5, 0)\}$

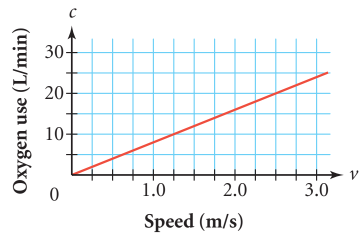
a. Find $g(f(4))$. **b.** Find $f(g(-2))$. **c.** Find $f(g(f(3)))$.

3. **APPLICATION** Graph A shows a swimmer's speed as a function of time. Graph B shows the swimmer's oxygen consumption as a function of her speed. Time is measured in seconds, speed in meters per second, and oxygen consumption in liters per minute. Use the graphs to estimate the values.

Graph A



Graph B



a. the swimmer's speed after 20 s of swimming

b. the swimmer's oxygen consumption at a swimming speed of 1.5 m/s

c. the swimmer's oxygen consumption after 40 s of swimming

4. Identify each equation as a composition of functions, a product of functions, or neither. If it is a composition or a product, then identify the two functions that combine to create the equation.

a. $y = 5\sqrt{3 + 2x}$

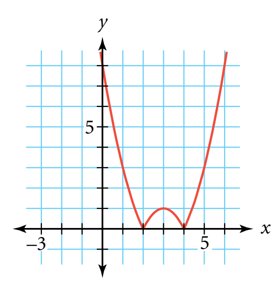
b. $y = 3 + \left(|x + 5| - 3\right)^2$

c. $y = (x - 5)^2\left(2 - \sqrt{x}\right)$

Keymath.com More Practice Your Skills

Reason and Apply

5. Consider the graph at right.



a. Write an equation for this graph.

b. Write two functions, f and g, such that the figure is the graph of $y = f(g(x))$.

6. The functions f and g are defined by these sets of input and output values.

$g = \{(1, 2), (-2, 4), (5, 5), (6, -2)\}$

$f = \{(2, 1), (4, -2), (5, 5), (-2, 6)\}$

a. Find $g(f(2))$.

b. Find $f(g(6))$.

c. Select any number from the domain of either g or f, and find $f(g(x))$ or $g(f(x))$, respectively. Describe what is happening.

7. A, B, and C are gauges with different linear measurement scales. When A measures 12, B measures 13, and when A measures 36, B measures 29. When B measures 20, C measures 57, and when B measures 32, C measures 84.

a. Sketch separate graphs for readings of B as a function of A and readings of C as a function of B. Label the axes.

b. If A reads 12, what does C read?

c. Write a function with the reading of B as the dependent variable and the reading of A as the independent variable.

d. Write a function with the reading of C as the dependent variable and the reading of B as the independent variable.

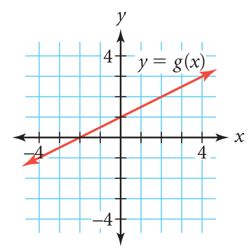
e. Write a function with the reading of C as the dependent variable and the reading of A as the independent variable.

8. The graph of the function $y = g(x)$ is shown at right. Draw a graph of each of these related functions.

a. $y = \sqrt{g(x)}$

b. $y = |g(x)|$

c. $y = (g(x))^2$



9. The two lines pictured at right are $f(x) = 2x - 1$ and $g(x) = \frac{1}{2}x + \frac{1}{2}$. Solve each problem both graphically and numerically.

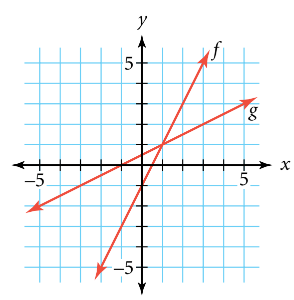
a. Find $g(f(2))$.

b. Find $f(g(-1))$.

c. Pick your own x-value in the domain of f, and find $g(f(x))$.

d. Pick your own x-value in the domain of g, and find $f(g(x))$.

e. Carefully describe what is happening in these compositions.



10. Given the functions $f(x) = -x^2 + 2x + 3$ and $g(x) = (x - 2)^2$, find these values.

a. $f(g(3))$ **b.** $f(g(2))$ **c.** $g(f(0.5))$ **d.** $g(f(1))$

e. $f(g(x))$. Simplify to remove all parentheses.

f. $g(f(x))$. Simplify to remove all parentheses.

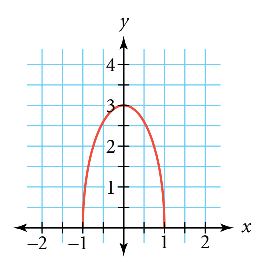
[▶ See **Calculator Note 4D** to learn how to use your calculator to check the answers to 10e and 10f. ◀]

11. Aaron and Davis need to write the equation that will produce the graph at right.

Aaron: "This is impossible! How are we supposed to know if the parent function is a parabola or a semicircle? If we don't know the parent function, there is no way to write the equation."

Davis: "Don't panic yet. I am sure we can determine its parent function if we study the graph carefully."

Do you agree with Davis? Explain completely and, if possible, write the equation of the graph.



12. **APPLICATION** Jen and Priya decide to go out to the Hamburger Shack for lunch. They each have a 50-cent coupon from the Sunday newspaper for the Super-Duper-Deluxe \$5.49 Value Meal. In addition, if they show their I.D. cards, they'll also get a 10% discount. Jen's server rang up the order as Value Meal, coupon, and then I.D. discount. Priya's server rang it up as Value Meal, I.D. discount, and then coupon.

a. How much did each girl pay?

b. Write a function, $C(x)$, that will deduct 50 cents from a price, x.

c. Write a function, $D(x)$, that will take 10% off a price, x.

d. Find $C(D(x))$.

e. Which server used $C(D(x))$ to calculate the price of the meal?

f. Is there a price for the Value Meal that would result in both girls paying the same price? If so, what is it?

Review

13. Solve.

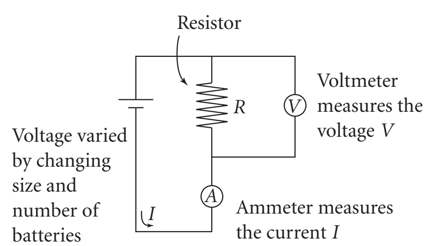
a. $\sqrt{|x-4|}=3$

b. $\left(3-\sqrt{x+2}\right)^2=4$

c. $\left|3-\sqrt{x}\right|=5$

d. $3+5\sqrt{1+2x^2}=13$

14. **APPLICATION** Bonnie and Mike are working on a physics project. They need to determine the ohm rating (resistance in ohms) of a resistor. Electrical resistance, measured in ohms, is defined as potential difference, measured in volts, divided by current, measured in amperes (amps). In their project they set up the circuit at right. They vary the potential difference shown on the voltmeter and observe the corresponding readings of current measured on the amp meter.



Voltage	12	10	6	4	3	1
Amps	2.8	2.1	1.4	1.0	0.6	0.2

a. Identify the independent and dependent variables.

b. Display the data on a graph.

c. Find the median-median line.

d. Bonnie and Mike reason that because 0 volts obviously yields 0 amps the line they really want is the median-median line translated to go through (0, 0).What is the equation of the line through the origin that is parallel to the median-median line?

e. How is the ohm rating Bonnie and Mike are trying to determine related to the line in 14d?

f. What is their best guess of the resistance to the nearest tenth of an ohm?

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15. Begin with the equation of the unit circle, $x^2 + y^2 = 1$.

a. Apply a horizontal stretch by a factor of 3 and a vertical stretch by a factor of 3, and write the equation that results.

b. Sketch the graph. Label the intercepts.

16. Imagine translating the graph of $f(x) = x^2$ left 3 units and up 5 units, and call the image $g(x)$.

a. Give the equation for $g(x)$.

b. What is the vertex of the graph of $y = g(x)$?

c. Give the coordinates of the image point that is 2 units to the right of the vertex.

Project

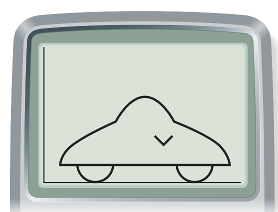
BOOLEAN GRAPHS

In his book *An Investigation into the Laws of Thought* (1854), the English mathematician George Boole (1815–1864) approached logic in a system that reduced it to simple algebra. In his system, later called Boolean algebra or symbolic logic, expressions are combined using "and" (multiplication), "or" (addition), and "not" (negative), and then interpreted as "true" (1) or "false" (0). Today, **Boolean algebra** plays a fundamental role in the design, construction, and programming of computers. You can learn more about Boolean algebra with the Internet links at www.keymath.com/DAA .

An example of a Boolean expression is $x \le 5$. In this case, if x is 10, the expression is false and assigned a value of 0. If x is 3, then the expression is true and it is assigned a value of 1. You can use Boolean expressions to limit the domain of a function when graphing on your calculator. For example, the graph of $Y_1 = (x + 4)/(x \le 5)$ does not exist for values of x greater than 5, because your calculator would be dividing by 0.

[▶ See **Calculator Note 4G** to learn more about graphing functions with Boolean expressions. ◀]

You can use your calculator to draw this car by entering the following short program.



```
PROGRAM:CAR
ClrDraw
DrawF 1/(X≥1)/(X≤9)
DrawF (1.2√(X-1)+1)/(X≤3.5)
DrawF (1.2√(-(X-9))+1)/(X≥6.5)
DrawF (-0.5(X-5)²+4)/(X≥3.5)/(X≤6.5)
DrawF -√(1-(X-2.5)²)+1
DrawF -√(1-(X-7.5)²)+1
DrawF (abs(X-5.5)+2)/(X≥5.2)/(X≤5.8)
```

Write your own program that uses functions, transformations, and Boolean expressions to draw a picture. Your project should include

- A screen capture or sketch of your drawing.
- The functions you used to create your drawing.

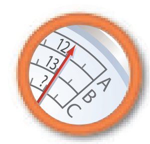
CHAPTER 4 REVIEW

Keymath.com Links to Resources

This chapter introduced the concept of a **function** and reviewed **function notation.** You saw real-world situations represented by rules, sets, functions, graphs, and most importantly, equations. You learned to distinguish between functions and other **relations** by using either the definition of a function—at most one y-value per x-value—or the vertical line test.

This chapter also introduced several **transformations,** including **translations, reflections,** and vertical and horizontal **stretches** and **shrinks.** You learned how to transform the graphs of **parent functions** to investigate several families of functions—linear, quadratic, square root, absolute value, and semicircle. For example, if you stretch the graph of the parent function $y = x^2$ by a factor of 3 vertically and by a factor of 2 horizontally, and translate it right 1 unit and up 4 units, then you get the graph of the function $y = 3\left(\frac{x-1}{2}\right)^2 + 4$.

Finally, you looked at the **composition** of functions. Many times, solving a problem involves two or more related functions. You can find the value of a composition of functions by using algebraic or numeric methods or by graphing.



EXERCISES

1. Sketch a graph that shows the relationship between the time in seconds after you start microwaving a bag of popcorn and the number of pops per second. Describe in words what your graph shows.

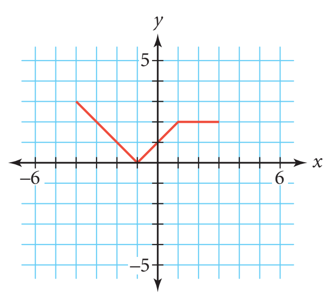
2. Use these three functions to find each value:

$f(x) = -2x + 7$
$g(x) = x^2 - 2$
$h(x) = (x + 1)^2$

a. $f(4)$ **b.** $g(-3)$ **c.** $h(x + 2) - 3$
d. $f(g(3))$ **e.** $g(h(-2))$ **f.** $h(f(-1))$
g. $f(g(a))$ **h.** $g(f(a))$ **i.** $h(f(a))$

3. The graph of $y = f(x)$ is shown at right. Sketch the graph of each of these functions:

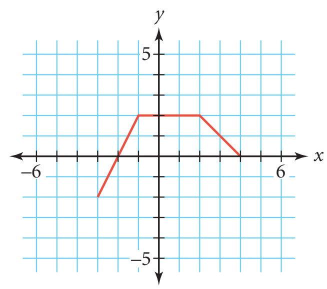
a. $y = f(x) - 3$
b. $y = f(x - 3)$
c. $y = 3f(x)$
d. $y = f(-x)$



4. Assume you know the graph of $y = f(x)$. Describe the transformations, in order, that would give you the graph of these functions:

a. $y = f(x + 2) - 3$ **b.** $\frac{y-1}{-1} = f\left(\frac{x}{2}\right) + 1$ **c.** $y = 2f\left(\frac{x-1}{0.5}\right) + 3$

5. The graph of $y = f(x)$ is shown at right. Use what you know about transformations to sketch these related functions:



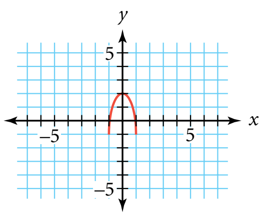
a. $y - 1 = f(x - 2)$ **b.** $\frac{y+3}{2} = f(x + 1)$

c. $y = f(-x) + 1$ **d.** $y + 2 = f\left(\frac{x}{2}\right)$

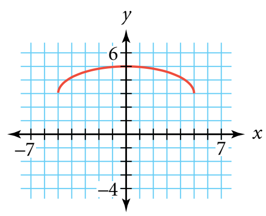
e. $y = -f(x - 3) + 1$ **f.** $\frac{y+2}{-2} = f\left(\frac{x-1}{1.5}\right)$

6. For each graph, name the parent function and write an equation of the graph.

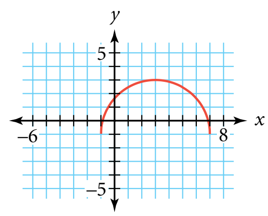
a.



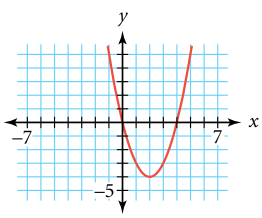
b.



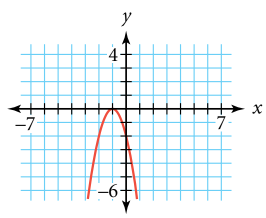
c.



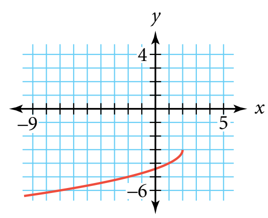
d.



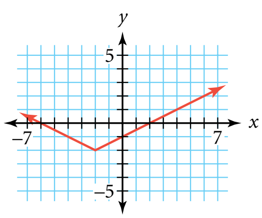
e.



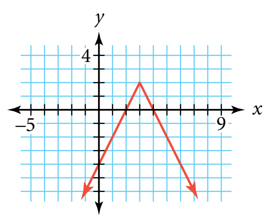
f.



g.



h.



7. Solve for y.

a. $2x - 3y = 6$ **b.** $(y + 1)^2 - 3 = x$ **c.** $\sqrt{1 - y^2} + 2 = x$

8. Solve for x.

a. $4\sqrt{x - 2} = 10$ **b.** $\left(\frac{x}{-3}\right)^2 = 5$ **c.** $\left|\frac{x-3}{2}\right| = 4$ **d.** $3\sqrt{1 + \left(\frac{x}{5}\right)^2} = 2$

9. The Acme Bus Company has a daily ridership of 18,000 passengers and charges \$1.00 per ride. The company wants to raise the fare yet keep its revenue as large as possible. (The revenue is found by multiplying the number of passengers by the fare charged.) From previous fare increases, the company estimates that for each increase of \$0.10 it will lose 1000 riders.

a. Complete this table.

Fare (\$) x	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80
Number of passengers	18000								
Revenue (\$) y	18000								

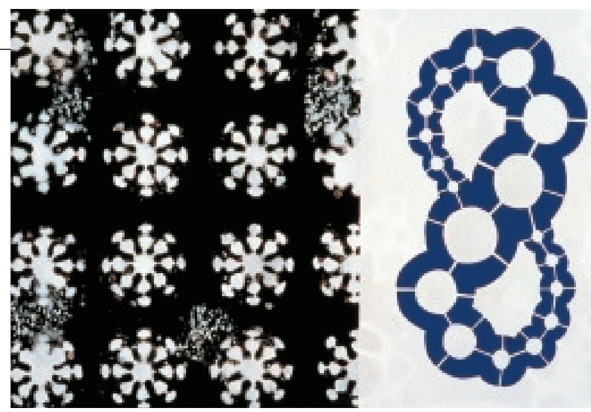
b. Make a graph of the the revenue versus fare charged. You should recognize the graph as a parabola.

c. What are the coordinates of the vertex of the parabola? Explain the meaning of each coordinate of the vertex.

d. Find a quadratic function that models these data. Use your model to find

i. the revenue if the fare is \$2.00.
ii. the fare(s) that make no revenue (\$0).

TAKE ANOTHER LOOK



This painting by Laura Domela is titled *sense* (2002, oil on birch panel). The design on the left is similar to an even function, while the one on the right is similar to an odd function.

1. Some functions can be described as even or odd. An **even function** has the y-axis as a line of symmetry. If the function f is an even function, then $f(-x) = f(x)$ for all values of x in the domain. Which parent functions that you've seen are even functions? Now graph $y = x^3$, $y = \frac{1}{x}$, and $y = \sqrt[3]{x}$, all of which are **odd functions.** Describe the symmetry displayed by these odd functions. How would you define an odd function in terms of $f(x)$? Can you give an example of a function that is neither even nor odd?

2. A line of reflection does not have to be the x- or y-axis. Draw the graph of a function and then draw its image when reflected across several different horizontal or vertical lines. Write the equation of each image. Try this with several different functions. In general, if the graph of $y = f(x)$ is reflected across the vertical line $x = a$, what is the equation of the image? If the graph of $y = f(x)$ is reflected across the horizontal line $y = b$, what is the equation of the image?

3. For the graph of the parent function $y = x^2$, you can think of any vertical stretch or shrink as an equivalent horizontal shrink or stretch. For example, the equations $y = 4x^2$ and $y = (2x)^2$ are equivalent, even though one represents a vertical stretch by a factor of 4 and the other represents a horizontal shrink by a factor of $\frac{1}{2}$. For the graph of any function or relation, is it possible to think of any vertical stretch or shrink as an equivalent horizontal shrink or stretch? If so, explain your reasoning. If not, give examples of functions and relations for which it is not possible.

4. Enter two linear functions into Y_1 and Y_2 on your calculator. Enter the compositions of the functions as $Y_3 = Y_1(Y_2(x))$ and $Y_4 = Y_2(Y_1(x))$. Graph Y_3 and Y_4 and look for any relationships between them. (It will help if you turn off the graphs of Y_1 and Y_2.) Make a conjecture about how the compositions of any two linear functions are related. Change the linear functions in Y_1 and Y_2 to test your conjecture. Can you algebraically prove your conjecture?

5. One way to visualize a composition of functions is to use a web graph. Here's how you evaluate $f(g(x))$ for any value of x, using a web graph:

Choose an x-value. Draw a vertical line from the x-axis to the function $g(x)$. Then draw a horizontal line from that point to the line $y = x$. Next, draw a vertical line from this point so that it intersects $f(x)$. Draw a horizontal line from the intersection point to the y-axis. The y-value at this point of intersection gives the value of $f(g(x))$.

Choose two functions $f(x)$ and $g(x)$. Use web graphs to find $f(g(x))$ for several values of x. Why does this method work?

Assessing What You've Learned

ORGANIZE YOUR NOTEBOOK Organize your notes on each type of parent function and each type of transformation you have learned about. Review how each transformation affects the graph of a function or relation and how the equation of the function or relation changes. You might want to create a large chart with rows for each type of transformation and columns for each type of parent function; don't forget to include a column for the general function, $y = f(x)$.

UPDATE YOUR PORTFOLIO Choose one piece of work that illustrates each transformation you have studied in this chapter. Try to select pieces that illustrate different parent functions. Add these to your portfolio. Describe each piece in a cover sheet, giving the objective, the result, and what you might have done differently.

WRITE TEST ITEMS Two important skills from this chapter are the ability to use transformations to write and graph equations. Write at least two test items that assess these skills. If you work with a group, identify other key ideas from this chapter and work together to write an entire test.

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