

## CHAPTER

# 8

# Parametric Equations and Trigonometry



Project Bandaloop is a group of contemporary dancers who combine dance, rock climbing, and rapelling. Their choreography relies upon a blend of aerial, horizontal, and vertical movement. By performing in a variety of venues, ranging from skyscrapers to granite cliffs, the group hopes to stimulate the viewer's awareness of his or her natural and human-made environment.

### OBJECTIVES

In this chapter you will

- Use a third variable to write parametric equations that separately define  $x$  and  $y$
- Simulate objects in motion with parametric equations
- Convert between parametric equations and equations that use only  $x$  and  $y$
- Review the trigonometric ratios—sine, cosine, and tangent
- Use the trigonometric ratios and their inverses to solve application problems

[Contents](#)[◀ Back](#)[Next ▶](#)[Index](#)

# LESSON

# 8.1

*Envisioning the end is enough  
to put the means in motion.*

DOROTHEA BRANDE

## Graphing Parametric Equations

**S**herlock Holmes followed footprints and other clues to track down suspected criminals. As he followed the clues, he knew exactly where the person had been. The path could be drawn on a map, every location described by  $x$ - and  $y$ -coordinates. But how could he determine *when* the suspect was at each place? Two variables are often not enough to describe interesting situations fully. You can use **parametric equations** to describe the  $x$ - and  $y$ -coordinates of a point separately as functions of a third variable,  $t$ , called the **parameter**. Parametric equations provide you with more information and better control over what points you plot.

In this example, the variable  $t$  represents time, and you will write parametric equations to simulate motion on your calculator screen. Observe how  $t$  controls the  $x$ - and  $y$ -values.

### EXAMPLE A

#### Environmental CONNECTION

Many oil spills occur when overcrowded seaports and poor weather cause oil tankers to collide in the ocean. Whether tankers are traveling toward land or to an offshore refinery to deliver oil, these conditions threaten the safety of less sturdy, single-hulled ships. Since the passing of the Oil Pollution Act of 1990, which enforced federal safety regulations on the transport of oil, tanker spills have been at an all-time low.

Two tankers leave Corpus Christi, Texas, at the same time, traveling toward St. Petersburg, Florida, 900 mi east. Tanker A travels at a constant speed of 18 mi/h, and Tanker B travels at a constant speed of 22 mi/h.

Write parametric equations and use your calculator to simulate the motion involved in this situation.

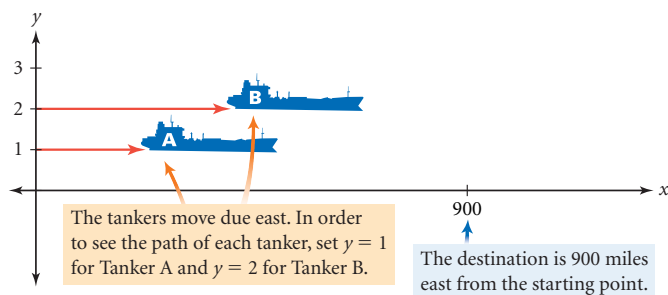


### ► Solution

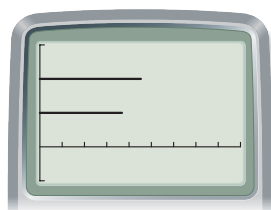
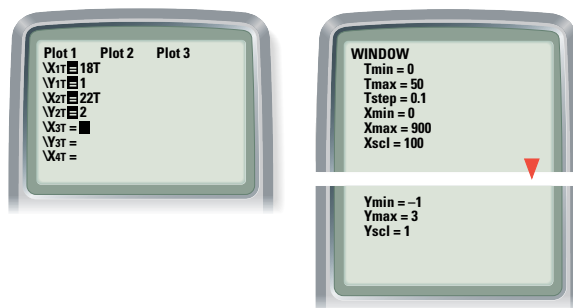
In order to graph the situation, you must first establish a coordinate system. Locate the origin,  $(0, 0)$ , at Corpus Christi. Because St. Petersburg is directly east, its coordinates are  $(900, 0)$ . The  $x$ -coordinate for each plotted point is the distance of a tanker from Corpus Christi, and the  $y$ -coordinate for each path will remain constant because the tankers travel directly east.

Set your calculator in Parametric and Simultaneous mode. See **Calculator Note 8A** to learn about the mode settings for parametric equations. To determine your equations, think about the motion of the tankers. After 1 h traveling at 18 mi/h, Tanker A will be 18 mi from Corpus Christi. After 2 h, it will be 36 mi away; after 10 h, it will be 180 mi away; and after  $t$  h, it will be  $18t$  mi away. The horizontal distance traveled, then, is determined by  $x = 18t$ . This equation

locates the horizontal position of Tanker A at any given time; the  $x$ -value is dependent on the time, or  $t$ -value. The tanker does not travel north or south at all, so the  $y$ -value must remain constant. If you choose a  $y$ -value of 0, the tanker will travel on the  $x$ -axis, and you won't be able to see it on the calculator. Instead, indicate this is the first tanker, Tanker A, by setting  $y = 1$ . The equations  $x = 18t$  and  $y = 1$  are a pair of parametric equations. You can model the motion of Tanker B with the parametric equations  $x = 22t$  and  $y = 2$ . Why is its horizontal position determined by  $22t$ ?

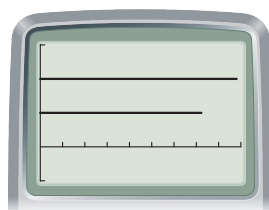


To find a good graphing window, consider the situation. The tankers must travel 900 mi east, so the  $x$ -values should be in the domain  $0 \leq x \leq 900$ . You chose constant  $y$ -values of 1 and 2, so the range  $-1 \leq y \leq 3$  is sufficient. When graphing parametric equations, you must also determine an interval for  $t$ . The slower tanker goes 18 miles per hour, so it takes 50 h to go 900 mi; therefore,  $t$ -values must range from 0 to 50. Enter these values into your calculator as shown below. Note that you must also enter a  $t$ -step. A  $t$ -step of 0.1 means that a point is plotted every 0.1 h (or 6 min). Graph the equations on your calculator and observe the motion of the tankers. ▶ See **Calculator Note 8B** to learn how to enter and graph parametric equations, and to learn more about the window settings. ◀



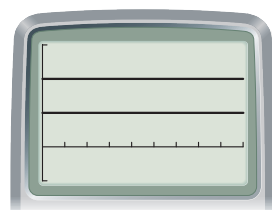
$[0, 900, 100, -1, 3, 1]$

The graph when  $t = 20$ .



$[0, 900, 100, -1, 3, 1]$

The graph when  $t = 40$ .



$[0, 900, 100, -1, 3, 1]$

The graph when  $t = 50$ .

When you model motion using parametric equations, you need to consider the object's direction as well as speed. Directed speed is called **velocity**. Velocity, unlike speed, can be either positive or negative. If you correctly determine whether the velocity is positive or negative, you will be able to see a realistic simulation of the motion on your calculator screen. The table at right shows how to relate the sign of the velocity to the direction of the motion.

| Motion        | Velocity |
|---------------|----------|
| Up or north   | Positive |
| Down or south | Negative |
| Left or west  | Negative |
| Right or east | Positive |



## Investigation Simulating Motion

In this investigation you will explore how to use parametric equations and graphs to answer questions about motion.

- Step 1** Enter the equations and graphing window from Example A into your calculator. Trace the path of the appropriate tanker, and solve equations to help you answer each question. [▶] See **Calculator Note 8C** to learn about tracing parametric equations. ◀]
- How long does it take the faster tanker to reach St. Petersburg?
  - Where is the slower tanker when the faster tanker reaches its destination?
  - When, during the trip, is the faster tanker exactly 82 mi in front of the slower tanker?
  - During what part of the trip are the tankers less than 60 mi apart?
- Step 2** Create another question involving these two tankers. Write an explanation and answer for your question on a separate sheet of paper. Exchange your question with another group and try to answer each other's questions.

Parametric equations allow you to graph paths when the location of points is dependent on time or some other parameter. A parametric representation lets you see the dynamic nature of the motion, and you can adjust the plotting speed by changing the  $t$ -step. The parameter,  $t$ , doesn't always represent time; it can be a unitless number. In the next example, you will control which  $x$ - and  $y$ -values are plotted by limiting the range of  $t$ , and you'll explore transformations of parametric functions.

### EXAMPLE B

Consider the parametric equations  $x = t$  and  $y = t^2$  for  $-1 \leq t \leq 2$ .

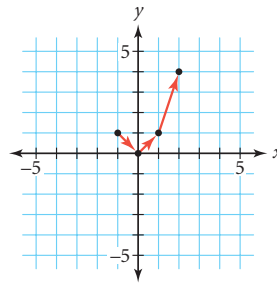
- Graph the equations on graph paper and on your calculator.
- Write equations to translate this parabola right 2 units and down 3 units.

### ► Solution

Use the equations to calculate  $x$ - and  $y$ -values that correspond to values of  $t$  in the range  $-1 \leq t \leq 2$ . If you use more  $t$ -values, you will have a smoother graph.

- Graph the points as  $t$  increases, connecting each point to the previous one, and add arrows to indicate the direction.

| $t$ | $x$ | $y$ |
|-----|-----|-----|
| -1  | -1  | 1   |
| 0   | 0   | 0   |
| 1   | 1   | 1   |
| 2   | 2   | 4   |

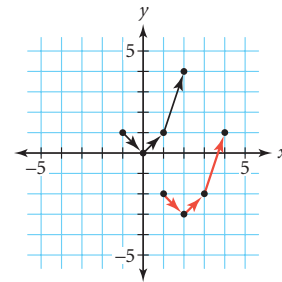


Verify this graph on your calculator.

- b. To translate the graph right 2 units, every  $x$ -coordinate should be increased by 2. To translate the graph down 3 units, every  $y$ -coordinate should be decreased by 3. The equations are

$$x = t + 2$$

$$y = t^2 - 3$$



A graph defined parametrically is the set of points  $(x, y)$  defined by  $t$ . What was the effect of limiting the values of the variable  $t$  on the parabola in Example B? What do you think the effect will be if  $t$ -values are limited to  $-2 \leq t \leq 1$ ? You will continue to explore translations and transformations of parametric functions in the exercises.

## EXERCISES

### Practice Your Skills

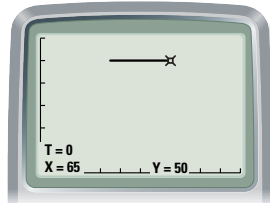
- Create a table for each equation with  $t = \{-2, -1, 0, 1, 2\}$ .
  - $x = 3t - 1$   
 $y = 2t + 1$
  - $x = t + 1$   
 $y = t^2$
  - $x = t^2$   
 $y = t + 3$
  - $x = t - 1$   
 $y = \sqrt{4 - t^2}$
- Graph each pair of parametric equations on your calculator. Sketch the result and use arrows to indicate the direction of increasing  $t$ -values along the graph. Limit your  $t$ -values as indicated, or, if an interval for  $t$  isn't listed, then find one that shows all of the graph that fits in a friendly window with a factor of 2.
  - $x = 3t - 1$   
 $y = 2t + 1$
  - $x = t + 1$   
 $y = t^2$
  - $x = t^2$   
 $y = t + 3$   
 $-2 \leq t \leq 1$
  - $x = t - 1$   
 $y = \sqrt{4 - t^2}$   
 $-2 \leq t \leq 2$
- Explore the graphs described in 3a-e, using a friendly window with a factor of 2 and  $-10 \leq t \leq 10$ .
  - Graph  $x = t$  and  $y = t^2$ .
  - Graph  $x = t + 2$  and  $y = t^2$ . How does this graph compare with the graph in 3a?
  - Graph  $x = t$  and  $y = t^2 - 3$ . How does this graph compare with the graph in 3a?

- d. Predict how the graph of  $x = t + 5$  and  $y = t^2 + 2$  compares with the graph in 3a. Verify your answer with a graph.
- e. Predict how the graph of  $x = t + a$  and  $y = t^2 + b$  compares with the graph in 3a.
4. Explore these graphs, using a friendly window with a factor of 2 and  $-10 \leq t \leq 10$ .
- Graph  $x = t$  and  $y = |t|$ .
  - Graph  $x = t - 1$  and  $y = |t| + 2$ . How does this graph compare with the graph in 4a?
  - Write a pair of parametric equations that will translate the graph in 4a left 4 units and down 3 units. Verify your answer with a graph.
  - Graph  $x = 2t$  and  $y = |t|$ . How does this graph compare with the graph in 4a?
  - Graph  $x = t$  and  $y = 3|t|$ . How does this graph compare with the graph in 4a?
  - Describe how the numbers 2, 3, and 4 in the equations  $x = t + 2$  and  $y = 3|t| - 4$  change the graph in 4a. Verify your answer with a graph.

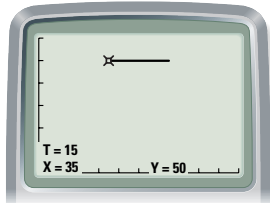


## Reason and Apply

5. The graphs below display a simulation of a team's mascot walking toward the west-end goal on the high school football field. Starting at a point 65 yards (yd) from the goal line and 50 ft from the sideline, she moves toward the goal line and waves at the crowd when she is 35 yd from the goal line. The parameter,  $t$ , represents time in seconds.



$[0, 100, 10, 0, 60, 10]$



$[0, 100, 10, 0, 60, 10]$

- According to the graphs, how much time has elapsed?
- How far did she travel?
- What is her average velocity?
- Explain the meaning of each number in these parametric equations.  

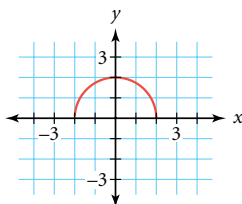
$$x = 65 - 2t$$

$$y = 50$$
- Using the equations from 5d, simulate the mascot's motion on your calculator screen in an appropriate window.
- Increase the maximum  $t$ -value in your graphing window to determine when the mascot crosses the 10-yard line.
- Write and solve an equation that will determine when the mascot crosses the 10-yard line.

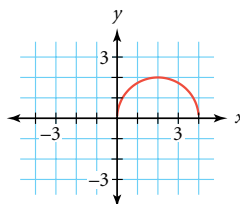


6. Write parametric equations for each graph. (*Hint: You can create parametric equations from a single equation that uses only  $x$  and  $y$  by letting  $x = t$  and changing  $y$  to a function of  $t$  by replacing  $x$  with  $t$ .)*

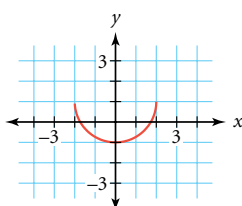
a.



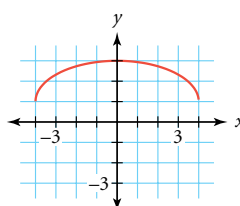
b.



c.

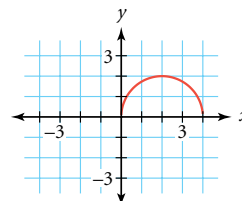


d.



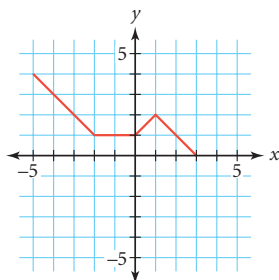
7. The graph of the parametric equations  $x = f(t)$  and  $y = g(t)$  is pictured at right.

- Sketch a graph of  $x = f(t)$  and  $y = -g(t)$  and describe the transformation.
- Sketch a graph of  $x = -f(t)$  and  $y = g(t)$  and describe the transformation.

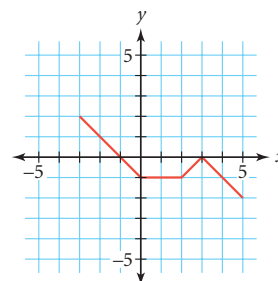
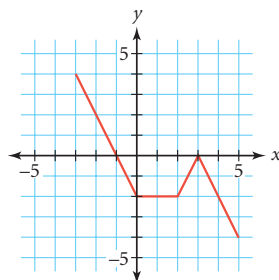


8. The graph of the parametric equations  $x = r(t)$  and  $y = s(t)$  is pictured at right. Write the parametric equations for each graph below in terms of  $r(t)$  and  $s(t)$ .

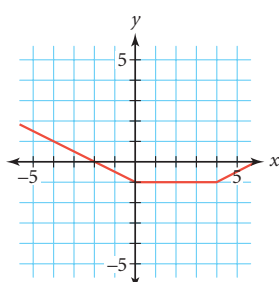
a.



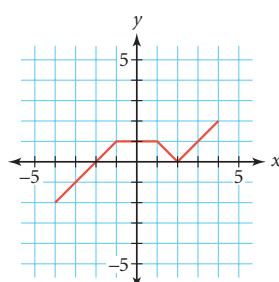
b.



c.



d.





9. Each spring a hare challenges a tortoise to a 50 m race. The hare knows that he can run faster than the tortoise, and he boasts that he can still win the race even if he gives the tortoise a 100 s head start. The tortoise crawls at a rate of 0.4 m/s, and the hare's running speed is 1.8 m/s.
- Write equations to simulate the motion of the tortoise.
  - Determine a graphing window and a range of  $t$ -values that will show the tortoise's motion.
  - Write equations to simulate the motion of the hare, who crosses the starting line at  $t = 100$ .
  - Who wins the race?
  - How long does it take each runner to finish the race?

10. Los Angeles, California, and Honolulu, Hawaii, are about 2500 mi apart. One plane flies from Los Angeles to Honolulu, and a second plane flies in the opposite direction.

- Describe the meaning of each number in the  $x$ -equations. (The equations for  $y$  simply assign noncolliding flight paths.)

$$\begin{array}{lcl} x = 450(t - 2) & \text{and} & x = 2500 - 525t \\ y = 1 & & y = 2 \end{array}$$

- Use tables or a graph to find the approximate time and location of the planes relative to Los Angeles when the planes pass each other.
- Write and solve an equation to find the time and location of the planes when they pass each other.



This illustration of the fable *The Tortoise and the Hare* is a wood engraving by French artist Gustave Doré (1832–1883).

### Career CONNECTION

Air traffic controllers coordinate airplane takeoffs and landings from airport control towers. They manage air traffic safety, track weather conditions, and communicate with pilots via radio. Sophisticated computer equipment and radar displays allow them to monitor airspace and prevent aircraft collisions. Air traffic control is considered one of the most demanding jobs in the United States. You can learn more about the mathematics required to become an air traffic controller with the Internet links at [www.keymath.com/DAA](http://www.keymath.com/DAA).



An airport control tower

11. These parametric equations simulate two walkers, with  $x$  and  $y$  measured in meters and  $t$  measured in seconds.

$$\begin{array}{lcl} x = 1.4t & \text{and} & x = 4.7 \\ y = 3.1 & & y = 1.2t \end{array}$$

- Graph their motion for  $0 \leq t \leq 5$ .
- Give real-world meanings for the values of 1.4, 3.1, 4.7, and 1.2 in the equations.
- Where do the two paths intersect?
- Do the walkers collide? How do you know this?



- 12. APPLICATION** An elephant stomps the ground creating sound waves in the air and vibrations (waves) in the ground. Sound waves travel through the air at approximately 0.3 km/s, and a vibration of the Earth's crust travels at approximately 6.1 km/s.
- Write parametric equations that model both waves for 3 s, and graph them on your calculator in an appropriate window.
  - How far does the wave in the ground travel in 3 s? How long does it take the sound wave in the air to travel that same distance?
  - A researcher's equipment detects the sound wave in the air 10 s after detecting the sound vibration in the ground. How far is the elephant from the researcher?

### Science

#### CONNECTION

American biologist and researcher Caitlin O'Connell-Rodwell proposes that elephants communicate with their feet. By stomping on the ground, elephants create seismic waves in the Earth's surface that can travel nearly 32 km—much farther than their communication by audible sound. Other elephants may detect these vibrations with their feet and interpret them as a warning of danger. Some researchers also believe elephants and other animals may detect earthquakes by hearing grinding rock and feeling tremors in the ground. Elephants may also be able to locate water by hearing thunderstorms and feeling the rumble of thunder through the ground.



Caitlin O'Connell-Rodwell listens to sounds that may be long-distance elephant communication.

### Review

- 13.** Solve this system.

$$\begin{cases} 3x + 5y = 6 \\ 4y = x - 19 \end{cases}$$

- 14.** Find the exact roots of this equation without using a calculator.

$$0 = x^3 - 4x^2 + 2x + 4$$

- 15.** Consider this sequence.

$$-6, -4, 3, 15, 32, \dots$$

- Find the  $n$ th term using finite differences. Assume  $-6$  is  $u_1$ .
  - Find  $u_{20}$ .
- 16.** Find the equation of the line through  $(-3, 10)$  and  $(6, -5)$ .
- 17.** Find the equation of the parabola that passes through the points  $(-2, -20)$ ,  $(2, 0)$ , and  $(4, -14)$ .

## LESSON

# 8.2

*An unhurried sense of time is  
in itself a form of wealth.*

BONNIE FRIEDMAN

# Converting from Parametric to Nonparametric Equations

In the previous lesson you saw parametric equivalents of functions you have studied earlier, like parabolas and absolute-value graphs. You can create these equations and graphs by letting  $x = t$  and making  $y$  a function of  $t$  instead of a function of  $x$ . For example, you can graph  $y = x^2$  in parametric form using  $x = t$  and  $y = t^2$ . The focus of this lesson is to do the reverse—to start with parametric equations and eliminate the parameter  $t$ , to get a single equation using only  $x$  and  $y$ .

In this investigation you will use parametric equations to model motion by measuring changes in both the  $x$ - and  $y$ -directions. You will discover the relationship between the graphs of the parametric equations, and the corresponding graph of the relation between  $x$  and  $y$ .



## Investigation Parametric Walk

### You will need

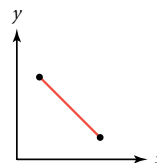
- two motion sensors
- masking tape

This investigation involves four participants: the walker, recorder X, recorder Y, and a director.

### Procedure Note

1. The walker starts at one end of the segment and walks slowly for 5 s to reach the other end.
2. Recorder X holds a motion sensor pointed at the walker, set for about 5 s, and moves along the  $y$ -axis keeping even with the walker, thus measuring the  $x$ -coordinate of the walker's path as a function of time.
3. Simultaneously, recorder Y holds a motion sensor pointed at the walker, set for the same amount of time, and moves along the  $x$ -axis keeping even with the walker, measuring the  $y$ -coordinate of the walker's path as a function of time.
4. The director starts all three participants at the same moment and counts out the 5 seconds.

- |        |   |
|--------|---|
| Step 1 | Mark a quadrant of a coordinate graph on the floor with tape identifying the $x$ - and $y$ -axes and a segment, as in the diagram at right.                     |
| Step 2 | Perform the activity as described in the procedure note. [▶] See <b>Calculator Note 8D</b> for additional instructions on how to set up your motion sensors. ◀] |
| Step 3 | Download the data from recorder X's motion sensor into a calculator, and find a function in the form $x = f(t)$ that fits the data.                             |
| Step 4 | Download the data from recorder Y's motion sensor into a different calculator, and find a function in the form $y = g(t)$ that fits the data.                   |



Step 5 On a third calculator, transfer the distance values from recorder X's motion sensor into list L1 and the distance values from recorder Y's motion sensor into list L2. Plot these data as ordered pairs  $(x, y)$  along with the parametric functions  $x = f(t)$  and  $y = g(t)$ . How do they compare?

Step 6 Solve  $x = f(t)$  for  $t$  and substitute this expression for  $t$  into  $y = g(t)$ .

Step 7 Graph your solution to Step 6 with the  $(x, y)$  data. ▶ See **Calculator Note 8E** to learn about drawing functions while still in Parametric mode. ◀

Step 8 Based on this investigation, explain what eliminating the parameter does to parametric equations.

During this investigation, two motion sensors captured data that you used to create a parametric model of a walk. You then merged these two graphs into a single graphic representation of the relation between  $x$  and  $y$ , and you merged the two parametric equations into a single equation using only  $x$  and  $y$ . You merge a pair of parametric equations into a single equation by eliminating the parameter  $t$ .

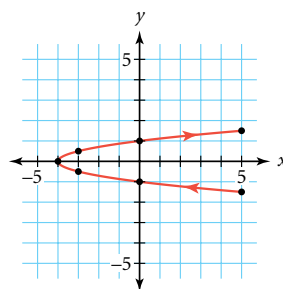
### EXAMPLE A

Graph the curve described by the parametric equations  $x = t^2 - 4$  and  $y = \frac{t}{2}$ . Then eliminate  $t$  from the equations and graph the result.

#### ► Solution

Make a table of values and plot the points, connecting them as  $t$  increases. Verify this graph on your calculator. Notice that the graph shows that  $y$  is not a function of  $x$ , even though both  $x$  and  $y$  are functions of  $t$ .

| $t$ | $x$ | $y$  |
|-----|-----|------|
| -3  | 5   | -1.5 |
| -2  | 0   | -1   |
| -1  | -3  | -0.5 |
| 0   | -4  | 0    |
| 1   | -3  | 0.5  |
| 2   | 0   | 1    |
| 3   | 5   | 1.5  |



To eliminate the parameter, solve one of the parametric equations for  $t$  and substitute into the other parametric equation.

$$x = t^2 - 4 \text{ and } y = \frac{t}{2}$$

$$t^2 = x + 4$$

$$t = \pm \sqrt{x + 4}$$

$$y = \frac{t}{2} = \frac{\pm \sqrt{x + 4}}{2}$$

$$y = \pm \frac{\sqrt{x + 4}}{2}$$

The parametric equations for  $x$  and  $y$ .

Add 4 to both sides in the parametric equation for  $x$ .

Take the square root of both sides to solve for  $t$ .

Substitute  $\pm \sqrt{x + 4}$  for  $t$  in the parametric equation for  $y$  to get a single equation using only  $x$  and  $y$ .

The single equation for  $y$  in terms of  $x$ .

You could also solve for  $t$  in the original parametric equation for  $y$  and substitute into the parametric equation for  $x$ .

$$x = t^2 - 4 \text{ and } y = \frac{t}{2}$$

The parametric equations for  $x$  and  $y$ .

$$t = 2y$$

Multiply by 2 to solve for  $t$  in the parametric equation for  $y$ .

$$x = (2y)^2 - 4$$

Substitute  $2y$  for  $t$  into the parametric equation for  $x$ .

$$x = 4y^2 - 4$$

Expand.

$$x + 4 = 4y^2$$

Add 4 to both sides.

$$\frac{x + 4}{4} = y^2$$

Divide both sides by 4.

$$y = \pm \sqrt{\frac{x + 4}{4}}$$

Take the square root of both sides.

$$y = \pm \frac{\sqrt{x + 4}}{2}$$

Take the square root of 4.

Notice that both methods result in the same equation.

You might recognize this as the equation of a horizontally oriented parabola, similar to equations you studied in Chapter 4. Check your result by graphing to show that the graphs of

$$y = \pm \frac{\sqrt{x + 4}}{2} \text{ and } \begin{aligned} x &= t^2 - 4 \\ y &= \frac{t}{2} \end{aligned}$$

are the same. [▶] Revisit **Calculator Note 8E** to learn how to simultaneously graph parametric equations and equations that use only  $x$  and  $y$ . ◀]

Although the same graph can often be created with parametric equations, or a single equation using only  $x$  and  $y$ , parametric equations show your position at particular times and allow you to graph relations that are not functions directly. The next example shows how two independent actions can combine to make a single path.

### EXAMPLE B

Hanna's hot-air balloon is ascending at a rate of 15 ft/s. A wind is blowing continuously from west to east at 24 ft/s. Write parametric equations to model this situation, and decide whether or not the hot-air balloon will clear power lines that are 300 ft to the east and 95 ft tall. Find the time it takes for the balloon to reach or pass over the power lines.



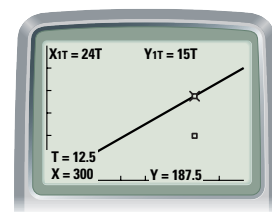
Hot-air balloons are made in all sorts of shapes.

► **Solution**

Create a table of time, ground distance, and height for a few seconds of flight. Set the origin as the initial launching location of the balloon. Let  $x$  represent the ground distance traveled to the east in feet, and let  $y$  represent the balloon's height above the ground in feet. The table below shows these values for the first 4 s of flight.

| Time (s)<br>$t$ | Ground distance (ft)<br>$x$ | Height (ft)<br>$y$ |
|-----------------|-----------------------------|--------------------|
| 0               | 0                           | 0                  |
| 1               | 24                          | 15                 |
| 2               | 48                          | 30                 |
| 3               | 72                          | 45                 |
| 4               | 96                          | 60                 |

The parametric equations that model the motion are  $x = 24t$  and  $y = 15t$ . Graph this pair of equations on your calculator. You can picture the power lines by plotting the point (300, 95). When you trace the graph to a time of 1 s, you will see that the balloon is 24 ft to the east, at a height of 15 ft. At 12.5 s, it has traveled 300 ft to the east and has reached a height of 187.5 ft. Hanna's balloon will not touch the power lines.



[0, 400, 100, 0, 300, 100]

Being able to model motion with parametric equations is much like the graphing you have done in earlier chapters, but you deal with each of the directions independently. This will often make difficult relations easier to model. Many pairs of parametric equations can be written as a single equation using only  $x$  and  $y$ . Being able to eliminate the parameter in parametric equations is an important skill because it gives you two different ways to study a relationship.

## EXERCISES

### Practice Your Skills

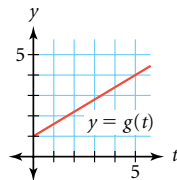
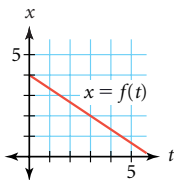
- Solve each equation for  $t$ .
  - $x = t + 1$
  - $x = 3t - 1$
  - $x = t^2$
  - $x = t - 1$
- Write each pair of parametric equations as a single equation using only  $x$  and  $y$ . Graph this new relation in a friendly graphing window. Verify that the graph of the new equation is the same as the graph of the pair of parametric equations.
  - $x = t + 1$   
 $y = t^2$
  - $x = 3t - 1$   
 $y = 2t + 1$
  - $x = t^2$   
 $y = t + 3$
  - $x = t - 1$   
 $y = \sqrt{4 - t^2}$
- Write a single equation (using only  $x$  and  $y$ ) that is equivalent to each pair of parametric equations.
  - $x = 2t - 3$   
 $y = t + 2$
  - $x = t^2$   
 $y = t + 1$
  - $x = \frac{1}{2}t + 1$   
 $y = \frac{t - 2}{3}$
  - $x = t - 3$   
 $y = 2(t - 1)^2$

4. The table at right gives  $x$ - and  $y$ -values for several values of  $t$ .

- Write an equation for  $x$  in terms of  $t$ .
- Write an equation for  $y$  in terms of  $t$ .
- Eliminate the parameter and combine the equations in 4a and 4b. Verify that this equation fits the values of  $(x, y)$ .

| $t$ | $x$ | $y$ |
|-----|-----|-----|
| 0   | 2   | 2   |
| 1   | 3   | 1   |
| 2   | 4   | 0   |
| 3   | 5   | -1  |
| 4   | 6   | 0   |
| 5   | 7   | 1   |
| 6   | 8   | 2   |

5. Use the graphs of  $x = f(t)$  and  $y = g(t)$  to create a graph of  $y$  as a function of  $x$ .



## Reason and Apply

- Write parametric equations for  $x = f(t)$  and  $y = g(t)$  given in Exercise 5, and an equation for  $y$  as a function of  $x$ . How do the slopes of these equations compare?
- Find the smallest interval for  $t$  that produces a graph of the parametric equations  $x = t + 2$  and  $y = t^2$  that just fits in a window with  $-5 \leq x \leq 5$  and  $-6 \leq y \leq 6$ .
- Consider the parametric equations  $x = f(t) = t + 2$  and  $y = g(t) = \sqrt{1 - t^2}$ .
  - Graph  $x = f(t)$  and  $y = g(t)$ .
  - Graph  $x = f(t)$  and  $y = -g(t)$  and identify the transformations of the original equations. Eliminate the parameter to write a single equation using only  $x$  and  $y$ .
  - Graph  $x = -f(t)$  and  $y = g(t)$  and identify the transformations of the original equations. Eliminate the parameter to write a single equation using only  $x$  and  $y$ .
  - Graph  $x = -f(t)$  and  $y = -g(t)$  and identify the transformations of the original equations. Eliminate the parameter to write a single equation using only  $x$  and  $y$ .

Titled *Tree-Lined Canal* (1971), this image of reflections was photographed in the Netherlands by Brett Weston.





9. A bug is crawling up a wall with locations given in the table at right. The variables  $x$  and  $y$  represent horizontal and vertical distances from the lower left corner of the wall, measured in inches, and  $t$  represents time measured in seconds.
- Write parametric equations for  $x$  and  $y$ , in terms of  $t$ , that generate the information given in the table.
  - Graph your parametric equations from 9a with the data points,  $(x, y)$ . How do they relate?
  - Using only  $x$  and  $y$ , write the equation of the line that passes through the data points.
  - How can the parametric equations be used to find the slope of the line in 9c?

| $t$ | $x$ | $y$ |
|-----|-----|-----|
| 0   | 20  | 5   |
| 2   | 24  | 7   |
| 4   | 28  | 9   |
| 6   | 32  | 11  |
| 8   | 36  | 13  |
| 10  | 40  | 15  |



In an area known for its many wood carvers, giant wooden bugs decorate the wall of this house on the Jersey Cove in Québec, Canada.

- Write parametric equations for two perpendicular lines that intersect at the point  $(3, 2)$ , with one line having a slope of  $-0.5$ .
- The functions  $d_1 = 1.5t$  and  $d_2 = 12 - 2.5t$  represent Edna's and Maria's distances in miles from a trailhead, as functions of time in hours.
  - Write parametric equations to simulate Edna's hike north away from the trailhead. Use  $x = 1$ .
  - Write parametric equations to simulate Maria's hike south toward the trailhead. Use  $x = 1.1$  so that Maria will come close to meeting Edna without actually bumping into her.
  - Name a graphing window and a range of  $t$ -values that show these hikes up to the moment the hikers meet.
  - Use your graph to approximate when and where the two hikers meet.
  - What equation can you write to find when Edna and Maria meet? Solve this equation symbolically to verify your answer to 11d.



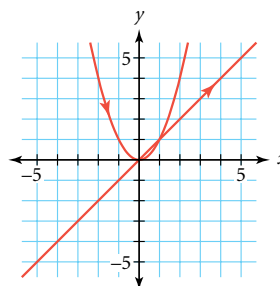
12. An egg is dropped from the roof of a 98 m building.
- How long will it take the egg to reach the ground?
  - Write and graph parametric equations to model the motion.
  - When will the egg be 1.75 m above the ground?
  - A 1.75 m high trampoline is rolled at a rate of 1.2 m/s toward the egg drop site, starting at the instant when the egg is dropped. How far away should the trampoline start so that when the egg is dropped it makes a direct hit?
  - Simulate both motions with parametric equations.

## Review

13. At right are the graphs of

$$\begin{array}{l} x = t \quad \text{and} \quad x = t \\ y = t \quad \text{and} \quad y = t^2 \end{array}$$

Write parametric equations of the parabola's reflection across the line  $y = x$ .



14. Tanker A moves at 18 mi/h and Tanker B moves at 22 mi/h. Both are traveling from Corpus Christi, Texas, to St. Petersburg, Florida, 900 mi directly east. Simulate the tanker movements if Tanker A leaves Corpus Christi at noon and Tanker B leaves at 5 P.M.

- Write parametric equations to simulate the motions.
- Name a graphing window and a range of  $t$ -values that show this situation.
- When and where does Tanker B pass Tanker A?
- Simulate the tanker movements if both tankers leave at noon, but Tanker A leaves from Corpus Christi and Tanker B leaves from St. Petersburg, each heading toward the other. Record your equations and determine the time interval during which they are within 50 mi of each other.

15. What is the equation of the image of  $y = \frac{2}{3}x - 2$  after a translation right 5 units and up 3 units?

16. Consider the function  $f(x) = 3 + \sqrt[3]{(x-1)^2}$ .

- a. Find

- $f(9)$
- $f(1)$
- $f(0)$
- $f(-7)$

- b. Find the equation(s) of the inverse of  $f(x)$ . Is the inverse a function?

- c. Describe how you can use your calculations in 16a to check your inverse in 16b.

- d. Use your calculator to graph  $f(x)$  and its inverse on the same axes.



An oil tanker travels in the Gulf of Mexico near Tampa, Florida.

# LESSON



# 8.3

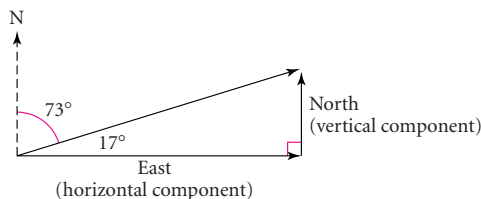
*Happiness is not a destination.  
It is a method of life.*  
BURTON HILLIS

## Right Triangle Trigonometry

Panama City is 750 miles from Corpus Christi, Texas, at a bearing of  $73^\circ$ . Model the movement of a tanker as it travels from Corpus Christi to Panama City, Florida.

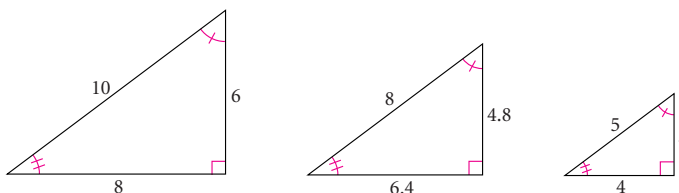


**T**rigonometry can be used with parametric equations to model motion that is not directly horizontal or vertical, such as the problem above. You can begin by separating any motion in two dimensions into its horizontal and vertical components. A **bearing** of  $73^\circ$  refers to a  $73^\circ$  angle measured clockwise from north, as shown below.

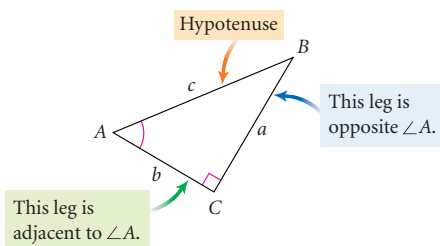


To solve a problem like this one, you will use **trigonometric ratios**.

The word **trigonometry** comes from the Greek words for “triangle” and “measure.” Trigonometry relates angle measures to ratios of sides in similar triangles. For example, in the similar right triangles shown below, all corresponding angles are congruent. The ratio of the length of the shorter leg to the length of the longer leg is always 0.75, and the ratios of the lengths of other pairs of corresponding sides are also equal. In right triangles, there are special names for each of these ratios.



## Trigonometric Ratios



For any acute angle  $A$  in a right triangle, the **sine** of  $\angle A$  is the ratio of the length of the leg opposite  $\angle A$  to the length of the hypotenuse.

$$\sin A = \frac{\text{opposite leg}}{\text{hypotenuse}} = \frac{a}{c}$$

The **cosine** of  $\angle A$  is the ratio of the length of the leg adjacent to  $\angle A$  to the length of the hypotenuse.

$$\cos A = \frac{\text{adjacent leg}}{\text{hypotenuse}} = \frac{b}{c}$$

The **tangent** of  $\angle A$  is the ratio of the length of the opposite leg to the length of the adjacent leg.

$$\tan A = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{a}{b}$$

### Language

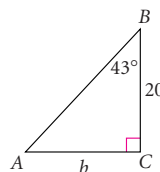
#### CONNECTION

The word *sine* has a curious history. The Sanskrit term for sine was *jya-ardha* (“half-chord”), later abbreviated to *jya*. Islamic scholars, who learned about sine from India, called it *jiba*. In Segovia, around 1140, Robert of Chester read *jiba* as *jaib* when he was translating al-Khwārizmī’s book, *Kitāb al-jabr wa’al-muqābalah*, from Arabic into Latin. One meaning of *jaib* is “indentation” or “gulf.” So *jiba* was translated into Latin as *sinus*, meaning “fold” or “indentation,” and from that we get the word *sine*.

You can use the trigonometric ratios to find unknown side lengths of a right triangle when you know the measure of one acute angle and the length of one of the sides. [▶ See **Calculator Note 8F** to learn about calculating the trigonometric ratios on your calculator. ◀]

### EXAMPLE A

Find the unknown length,  $b$ .



► **Solution**

In this problem, you know the length of one leg, you know the measure of one acute angle, and you want to find the length of the other leg. The tangent ratio relates the lengths of the legs to the measure of the angle. Therefore, you can use the tangent ratio. (It would be more difficult to use sine or cosine because you would need to write an expression for the length of the hypotenuse.)

$$\tan 43^\circ = \frac{b}{20}$$

Write the tangent ratio and substitute the known values of the angle measure and length of the adjacent leg.

$$b = 20 \tan 43^\circ$$

Multiply both sides by 20.

$$b \approx 18.65$$

Find the tangent of  $43^\circ$ , and multiply it by 20.

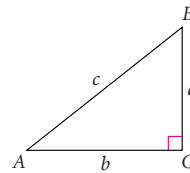
The length,  $b$ , of side  $\overline{AC}$  is approximately 18.65 units.

Note that the trigonometric ratios apply to both acute angles in a right triangle. In this triangle,  $b$  is the leg adjacent to  $A$  and the leg opposite  $B$ , whereas  $a$  is the leg opposite  $A$  and the leg adjacent to  $B$ . As always,  $c$  is the hypotenuse.

$$\sin A = \frac{a}{c} \quad \sin B = \frac{b}{c}$$

$$\cos A = \frac{b}{c} \quad \cos B = \frac{a}{c}$$

$$\tan A = \frac{a}{b} \quad \tan B = \frac{b}{a}$$



## Investigation Two Ships

In this investigation you will model the motion of two cargo ships traveling from Corpus Christi. Ship A is traveling on a bearing of  $73^\circ$  toward Panama City, 750 mi away. Ship B is traveling on a bearing of  $90^\circ$  toward St. Petersburg, 900 mi away. Both ships are traveling at 23 mi/h.



- Step 1 For each ship, write an equation that gives its distance from Corpus Christi as a function of time.
- Step 2 Solve each equation from Step 1 to find out when each ship will arrive at its destination.

As with other problems in this chapter, you will often need to choose a coordinate system or point of reference for application problems. Most of the time, it makes sense to locate the initial point of the problem at the origin.

- |        |   |
|--------|---|
| Step 3 | How far east of Corpus Christi is each ship after 1 h? After 2 h? How far north is each ship at those times? ( <i>Hint:</i> You may need to use trigonometry for Ship A.)   |
| Step 4 | Write a pair of parametric equations for each ship's location as a function of time. Explain where each number in each equation comes from. Change numbers that are approximations back to exact form (using trigonometric ratios). |
| Step 5 | How far north of Corpus Christi is Panama City? How far east?   |
| Step 6 | Use the parametric equations you wrote in Step 4 to find out when each ship will arrive at its destination. Explain how your answers relate to your answers from Step 2.  |

Each trigonometric ratio is a function of an acute angle measure, because each angle measure has a unique ratio associated with it. The inverse of each trigonometric function gives the measure of the angle. For example,  $\tan^{-1}\left(\frac{18}{20}\right) \approx 42^\circ$ .

### EXAMPLE B

Two hikers leave their campsite. One walks east 2.85 km and the other walks south 6.03 km.


- After the hikers get to their destinations, what is the bearing from the southern hiker to the eastern hiker?
- How far apart are they?

### ► Solution

Draw a diagram using the information given.

- Angle  $S$  measures the bearing from the southern hiker to the eastern hiker. This is a right triangle and you know the lengths of both legs, so you can use the tangent ratio.

$$\tan S = \frac{2.85}{6.03}$$

Take the inverse tangent of both sides to find the angle measure. Since you are composing tangent with its inverse,  $\tan^{-1}(\tan S)$  is equivalent to  $S$ .  Revisit **Calculator Note 8F** to learn about calculating the inverse trigonometric functions. ◀

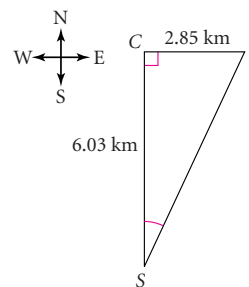
$$S = \tan^{-1}\left(\frac{2.85}{6.03}\right) \approx 25^\circ$$

The eastern hiker is at a bearing of about  $25^\circ$  from the southern hiker.

- You can use the Pythagorean Theorem to find the distance from  $S$  to  $E$ .

$$\begin{aligned} 6.03^2 + 2.85^2 &= (SE)^2 \\ SE &= \sqrt{6.03^2 + 2.85^2} \\ SE &\approx 6.67 \end{aligned}$$

The distance between the two hikers is approximately 6.67 km.





## Recreation CONNECTION

A compass is a useful tool to have if you're hiking, camping, or sailing in an unfamiliar setting. The circular edge of a compass has 360 marks representing degrees of direction. North is  $0^\circ$  or  $360^\circ$ , east is  $90^\circ$ , south is  $180^\circ$ , and west is  $270^\circ$ . Each degree on a compass shows the direction of travel, or bearing. The needle inside a compass always points north. Magnetic poles on the Earth guide the direction of the needle and allow you to navigate along your desired course. Learn how to use a compass with the links at [www.keymath.com/DAA](http://www.keymath.com/DAA).



Your calculator will display each trigonometric ratio to many digits, so your final answer could be displayed to several decimal places. However, it is usually appropriate to round your final answer to the nearest degree or 0.1 unit of length, as in the solution to part a in Example B. If the problem gives more precise measurements, you can use that same amount of precision in your answer. In the example, distances were given to the nearest 0.01 km, so the answer in part b was rounded to the nearest 0.01 also.

## EXERCISES

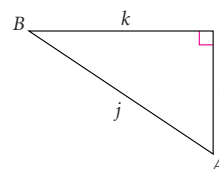
You will need



Geometry software  
for Exercise 16

### Practice Your Skills

- Write all the trigonometric formulas (including inverses) relating the sides and angles in this triangle. There should be a total of 12.



- Draw a right triangle for each problem. Label the sides and angle, then solve to find the unknown measure.

a.  $\sin 20^\circ = \frac{a}{12}$

b.  $\cos 80^\circ = \frac{25}{b}$

c.  $\tan 55^\circ = \frac{c+4}{c}$

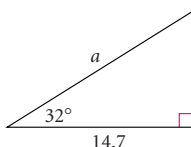
d.  $\sin^{-1} \frac{17}{30} = D$



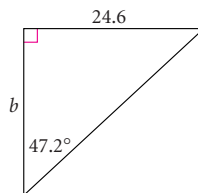
### Reason and Apply

- For each triangle, find the length of the labeled side.

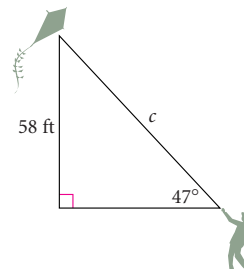
a.



b.

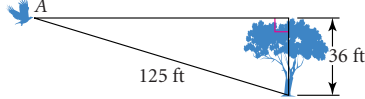


c.

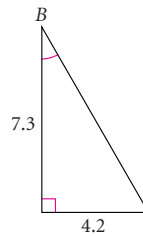


4. For each triangle, find the measure of the labeled angle.

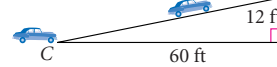
a.



b.



c.



5. Draw a pair of intersecting horizontal and vertical lines, and label north, east, south, and west. Sketch the path of a plane flying on a bearing of  $30^\circ$ .

- What is the measure of the angle between the plane's path and the horizontal axis?
- Choose any point along the plane's path. From this point, draw a segment perpendicular to the horizontal axis to create a right triangle. Label the length of the horizontal leg,  $x$ , and the vertical leg,  $y$ .
- The plane flies 180 mi/h for 2 h. How far east and how far north does it travel?

6. Consider the parametric equations

$$x = t \cos 39^\circ$$

$$y = t \sin 39^\circ$$

- Make a graph with window  $-1 \leq x \leq 10$ ,  $-1 \leq y \leq 10$ , and  $0 \leq t \leq 10$ .
- Describe the graph. What happens if you change the minimum and maximum values of  $t$ ?
- Find the measure of the angle between this line and the  $x$ -axis. (*Hint:* Trace to a point on the line and find the coordinates.)

7. Consider the parametric equations

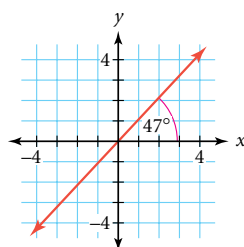
$$x = 5t \cos 40^\circ$$

$$y = 5t \sin 40^\circ$$

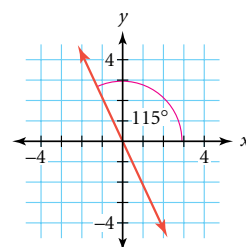
- Make a graph with window  $0 \leq x \leq 5$ ,  $-2 \leq y \leq 5$ , and  $0 \leq t \leq 1$ .
- Describe the graph and the measure of the angle between the graph and the  $x$ -axis.
- What is the relationship between this angle and the parametric equations?
- What is the effect of the 5 in each equation? Change the 5 to a 1 in the first equation. What happens? What happens when you change 5 to 1 in both equations?

8. Write parametric equations for each graph.

a.



b.



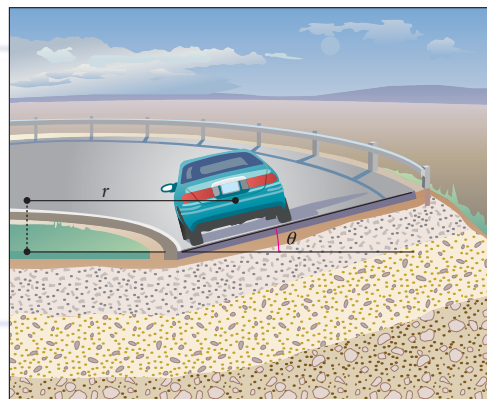
9. A plane is flying at 100 mi/h on a bearing of  $60^\circ$ .
- Draw a diagram of the motion. Draw a segment perpendicular to the x-axis to create a right triangle. Write equations for  $x$  and  $y$ , in terms of  $t$ , to model the motion.
  - What range of  $t$  is required to display 500 mi of plane travel? (Assume  $t$  represents time in hours.)
  - Explain the real-world meaning of the numbers and variables you used in your equations.
10. Tanker A is moving at a speed of 18 mi/h from Corpus Christi, Texas, toward Panama City, Florida. Panama City is 750 mi from Corpus Christi at a bearing of  $73^\circ$ .
- Make a sketch of the tanker's motion, including coordinate axes.
  - How long does the tanker take to get to Panama City?
  - How far east and how far north is Panama City from Corpus Christi?
11. Tanker B is traveling at a speed of 22 mi/h from St. Petersburg, Florida, to New Orleans, Louisiana, on a bearing of  $285^\circ$ . The distance between the two ports is 510 mi.
- Make a sketch of the tanker's motion, including coordinate axes.
  - How long will it take the tanker to get to New Orleans?
  - How far west and how far north is New Orleans from St. Petersburg?
  - Suppose Tanker A in Exercise 10 leaves at the same time as Tanker B. Describe where the ships' paths intersect. Recall that St. Petersburg is 900 miles east of Corpus Christi. Will the ships collide? Explain your answer.
12. **APPLICATION** Civil engineers generally bank, or angle, a curve on a road so that a car going around the curve at the recommended speed does not skid off the road. Engineers use this formula to calculate the proper banking angle,  $\theta$ , where  $v$  represents the velocity in meters per second,  $r$  represents the radius of the curve in meters, and  $g$  represents the gravitational constant,  $9.8 \text{ m/s}^2$ .
- $$\tan \theta = \frac{v^2}{rg}$$
- If the radius of an exit ramp is 60 m and the recommended speed is 40 km/h, at what angle should the curve be banked?
  - A curve on a racetrack is banked at  $36^\circ$ . The radius of the curve is about 1.7 km. What speed is this curve designed for?

### Science

#### CONNECTION

When a car rounds a curve, the driver must rely on the friction between the car's tires and the road surface to stay on the road. Unfortunately, this does not always work—especially if the road surface is wet!

In car racing, where cars travel at high speeds, tracks banked steeply allow cars to go faster, especially around the corners. Banking on NASCAR tracks ranges from  $36^\circ$  in the corners to just a slight degree of banking in the straighter portions.

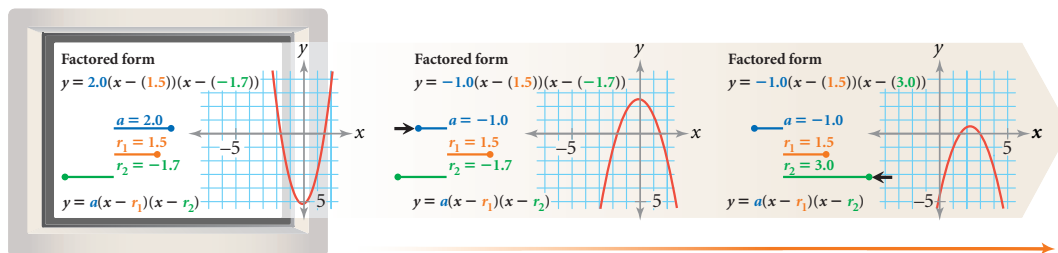


## Review

13. This table gives the position of a walker at several times.

| Time (min)<br>$t$ | Horizontal distance (m)<br>$x$ | Vertical distance (m)<br>$y$ |
|-------------------|--------------------------------|------------------------------|
| 0                 | 6                              | 5                            |
| 2                 | 14                             | 11                           |
| 4                 | 22                             | 17                           |
| 6                 | 30                             | 23                           |
| 8                 | 38                             | 29                           |
| 10                | 46                             | 35                           |

- Write a single equation for  $y$  in terms of  $x$  that fits the data points,  $(x, y)$ , listed in the table.
  - Parametric equations modeling this table are  $x = 6 + 4t$  and  $y = 5 + 3t$ . Eliminate the parameter,  $t$ , from these equations and compare your final equation to the answer you found in 13a.
14. Graph the parabola  $y = 35 - 4.9(x - 3.2)^2$ .
- What are the coordinates of the vertex?
  - What are the  $x$ -intercepts?
  - Where does the parabola intersect the line  $y = 15$ ?
15. Write the equation of the circle with center  $(2.6, -4.5)$  and radius 3.6.
16. **Technology** With geometry software you can create sliders—changeable line segments whose lengths represent the values of variables. The sliders can then be used to dynamically change the equation and graph of a function.
- Create a sketch that uses sliders for  $a$ ,  $h$ , and  $k$ , and shows how these parameters affect the vertex form of a quadratic equation,  $y = a(x - h)^2 + k$ .
  - Create another sketch that uses sliders for  $a$ ,  $r_1$ , and  $r_2$ , and shows how these parameters affect the factored form of a quadratic equation,  $y = a(x - r_1)(x - r_2)$ .



## EXPLORATION

# Parametric Equations for a Circle

By definition, a **circle** is the set of all the points in a plane at a given **radius** from a given **center** in the plane. To graph a complete circle on your calculator, you have used two separate functions: one for the top half of the circle and one for the bottom half. Graphing a circle is much simpler with parametric equations.

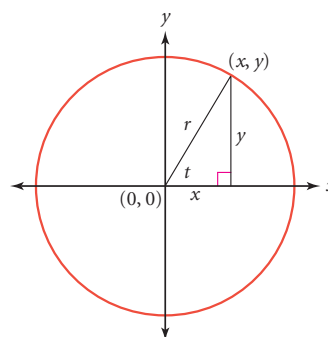
Imagine a circle with radius  $r$  and center at the origin. For a right triangle in the first quadrant with central angle measuring  $t$  degrees,  $\sin t = \frac{y}{r}$  and  $\cos t = \frac{x}{r}$ . Solving these equations for  $x$  and  $y$ , you get the parametric equations for a circle:

$$x = r \cos t$$

$$y = r \sin t$$

How do these equations work when the angles are more than  $90^\circ$ ? You can still apply the definitions of the trigonometric ratios.

However, when the angle is greater than  $90^\circ$ , you must use a right triangle situated in another quadrant. In each case, you form the triangle by connecting the point on the circle to the  $x$ -axis. For example, if the angle is  $150^\circ$ , you form a triangle where the angle with the  $x$ -axis is  $30^\circ$ . To find the values of the trigonometric ratios for  $150^\circ$ , you use the newly formed right triangle. Using a circle allows you to extend the definitions of sine, cosine, and tangent for any real angle value.



$$\sin t = \frac{y}{r}$$

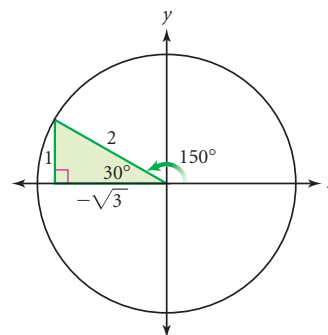
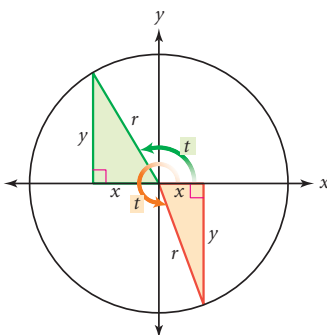
$$\cos t = \frac{x}{r}$$

$$\tan t = \frac{y}{x}$$

$$\sin 150^\circ = \frac{1}{2}$$

$$\cos 150^\circ = -\frac{\sqrt{3}}{2}$$

$$\tan 150^\circ = -\frac{1}{\sqrt{3}}$$



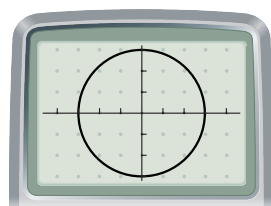
**EXAMPLE** Use parametric equations to graph the circle with radius 3 and center (0, 0).

► **Solution** The parametric equations for a circle with radius 3 and center (0, 0) are

$$x = 3 \cos t$$

$$y = 3 \sin t$$

The variable  $t$  represents the central angle of the circle. Use the range  $0^\circ \leq t \leq 360^\circ$  with a  $t$ -step of  $1^\circ$  to graph the complete circle on your calculator.



$[-4.7, 4.7, 1, -3.1, 3.1, 1]$

The  $t$ -step tells your calculator how often to plot a point. The calculator then connects these points with straight segments. When you make the  $t$ -step larger, the points are farther apart on the circle and your calculator's graph stops looking like a circle. This gives you a way to draw some interesting geometric shapes on your calculator.



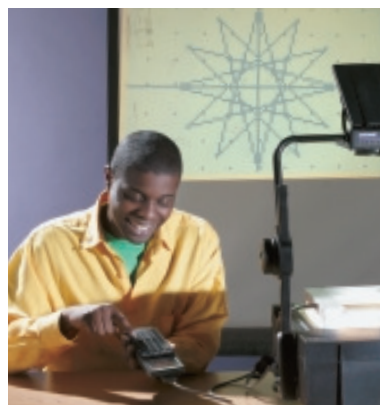
The Star of Hisham from the Umayyad Dynasty (660–750 C.E.), located in the Khirbat al-Mafjar palace, Israel, uses circles and a star-like shape inscribed within the larger circle.

## Activity

### Variations on a Circle

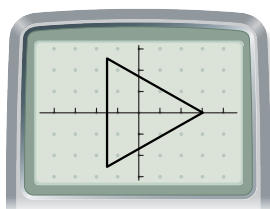
In this activity you'll explore what happens to the graph of a circle when you change the parameter  $t$ . Work individually on Step 1. Then work with a partner or group for the remainder of the activity.

- Step 1 Start with the parametric equations  $x = 3 \cos t$  and  $y = 3 \sin t$ . Experiment with the parameter  $t$ , changing the minimum and maximum values and the  $t$ -step. Use these questions to guide your exploration.

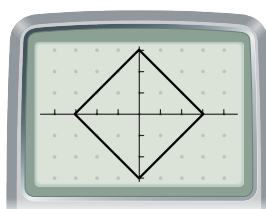




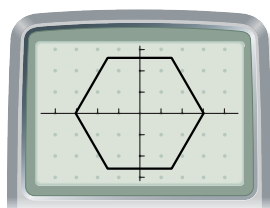
- What effect does changing the range of  $t$  have on the graph?
- What effect does changing the  $t$ -step have on the graph?
- How can you make a triangle? A square? A hexagon? An octagon? Try to find more than one way to draw each figure.



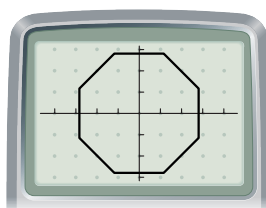
$[-4.7, 4.7, 1, -3.1, 3.1, 1]$



$[-4.7, 4.7, 1, -3.1, 3.1, 1]$



$[-4.7, 4.7, 1, -3.1, 3.1, 1]$



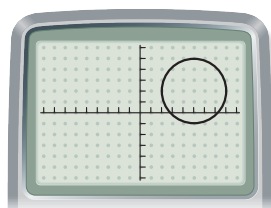
$[-4.7, 4.7, 1, -3.1, 3.1, 1]$

- How can you make a square with sides parallel to the axes?
- How can you rotate a polygon shape about the origin by altering the parametric equations?

**Step 2** Share your results from Step 1 with a partner or your group. Work together to write a paragraph summarizing your discoveries.

**Step 3** Find a way to translate the circle  $x = 3 \cos t$  and  $y = 3 \sin t$  so that it is centered at  $(5, 2)$ .

**Step 4** Reflect the graph from Step 3 across the  $y$ -axis, across the  $x$ -axis, across the line  $y = x$ , and across the line  $x = -1$ . Describe the method you used for each of these reflections.



$[-9.4, 9.4, 1, -6.2, 6.2, 1]$

**Step 5** Graph the equations  $x = 3 \cos t$  and  $y = 3 \sin t$ , using the range  $0^\circ \leq t \leq 3600^\circ$  with  $t$ -step  $125^\circ$ . Then try other  $t$ -step values that are not factors of  $360^\circ$ , such as  $100^\circ$ ,  $150^\circ$ , and  $185^\circ$ . (You may also need to increase the maximum value of  $t$ .) Write a paragraph explaining what happens in each case.

## Art

### CONNECTION

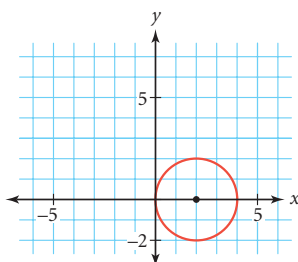
By about 2800 B.C.E., gemstone cutting was a widespread practice throughout Egypt and Asia Minor (modern Turkey). Today's gem cutters use the same geometric designs that ancient gem cutters used. In multiples of 8, they carve small symmetric planes on a gem, called facets. Rubies, sapphires, and emeralds are often square or rectangular in shape, whereas other gems are cut into triangular, diamond, or trapezoidal shapes. The oldest gemstone shape, however, is circular or rounded, which is how many opal and opaque gems are still cut today.



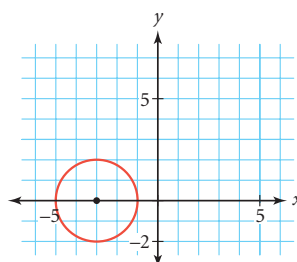
## Questions

- In the investigation you used the parametric equations for a circle,  $x = 3 \cos t$  and  $y = 3 \sin t$ , to graph other geometric shapes on your calculator. Mathematically, do you think it is correct to say that  $x = 3 \cos t$  and  $y = 3 \sin t$  are the parametric equations of a square or a hexagon? Explain your reasoning.
- Find parametric equations for each translated circle.

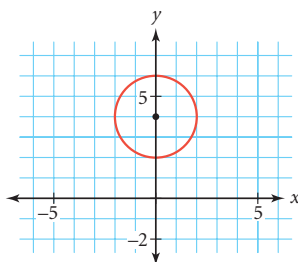
a.



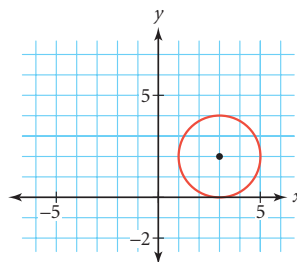
b.



c.

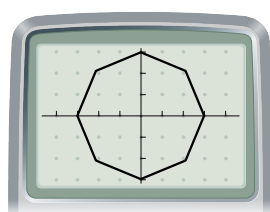


d.



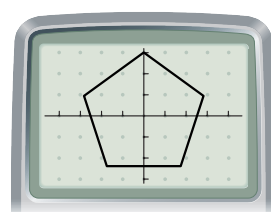
- Use the parametric equations  $x = 3 \cos t$  and  $y = 3 \sin t$  to make each of these figures on your calculator. What range of  $t$ -values and what  $t$ -step are required for each?

a.



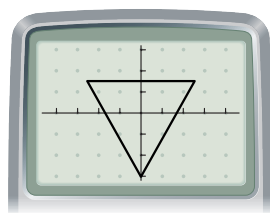
$[-4.7, 4.7, 1, -3.1, 3.1, 1]$

b.



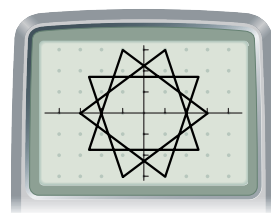
$[-4.7, 4.7, 1, -3.1, 3.1, 1]$

c.



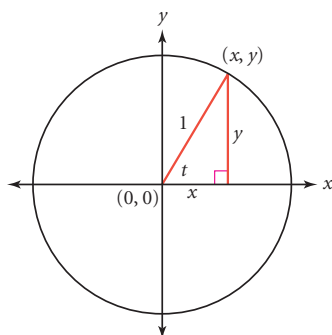
$[-4.7, 4.7, 1, -3.1, 3.1, 1]$

d.



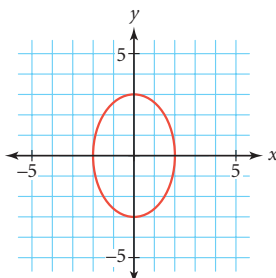
$[-4.7, 4.7, 1, -3.1, 3.1, 1]$

4. Consider a unit circle (a circle with radius 1) centered at the origin.

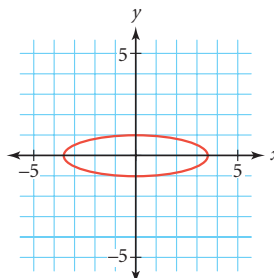


- What are the parametric equations for the unit circle?
  - From Chapter 4, you know that the Pythagorean Theorem yields the equation of the unit circle,  $x^2 + y^2 = 1$ . Substitute the parametric equations for  $x$  and  $y$  to get an equation in terms of sine and cosine.
  - Use your calculator to verify that your equation in 4b is true for  $t = 47^\circ$ .
5. Experiment to find parametric equations for each ellipse.

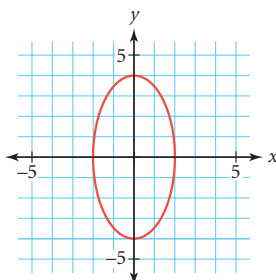
a.



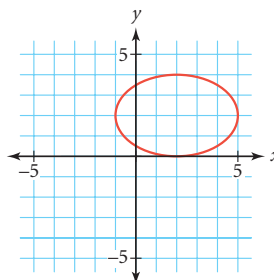
b.



c.



d.



# LESSON

## 8.4



*Not snow, no, nor rain, nor heat,  
nor night keeps them from  
accomplishing their appointed  
courses with all speed.*

HERODOTUS

# Using Trigonometry to Set a Course

**W**inds and air currents affect the direction and speed a plane will travel. A pilot flying an airplane has to compensate for these effects. Similarly, as you swim across a river a current can sweep you downstream, so that when you reach the other side you are not directly across from the point where you began. You can use parametric equations to model and simulate motion affected by a combination of forces.



In 1926, Gertrude Ederle (b 1906) of the United States swam across the 21-mile-wide English Channel in 14 h 31 min, breaking the previous male record by more than 2 h and becoming the first female to complete the challenge. Gale-force winds, rough waters, and strong currents made the swim nearly impossible.

This investigation simulates what happens when you try to swim or fly with, against, or across a strong water or air current.



## Investigation Motion in a Current

Asuka, Ben, and Chelsea are playing with remote-controlled boats in a pool. Each boat is moving 4 ft/s in the direction indicated. All the boats start moving at the same time.

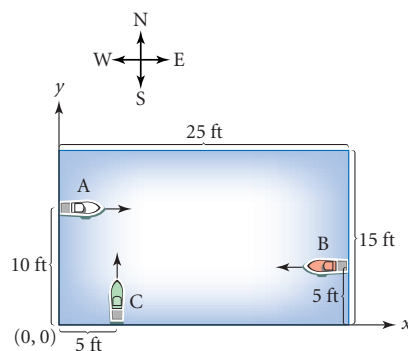
Step 1

Create a table like the one below showing the  $x$ - and  $y$ -positions of the three boats for 3 s.

| Time (s) | Boat A |     | Boat B |     | Boat C |     |
|----------|--------|-----|--------|-----|--------|-----|
| $t$      | $x$    | $y$ | $x$    | $y$ | $x$    | $y$ |
| 0        |        |     |        |     |        |     |
| 1        |        |     |        |     |        |     |
| 2        |        |     |        |     |        |     |
| 3        |        |     |        |     |        |     |

Step 2

Write and graph parametric equations to model the motion of the three boats. Identify the window and range of  $t$ -values you use to graph the simulation.

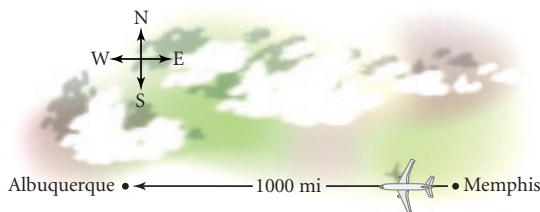


- |        |   |
|--------|---|
| Step 3 | Asuka, Ben, and Chelsea return their boats to the starting positions and repeat the motion, but a gust of wind starts blowing from west to east at a constant rate of 3 ft/s. Create another table of the positions of the boats over 3 s.  |
| Step 4 | Write and graph parametric equations to model the motion of the boats in Step 3. Show how to write each of these using the $4t$ from the boat's speed and the $3t$ from the wind's speed. Identify the window and range of $t$ -values you use to graph the simulation.   |
| Step 5 | Use the Pythagorean Theorem to determine the distance Boat C travels in 3 s, and its velocity.  |
| Step 6 | At what angle, $A$ , does Boat C travel?  |
| Step 7 | Use the velocity and angle you found in Steps 5 and 6 to write parametric equations in the form $x = x_0 + vt \cos A$ and $y = y_0 + vt \sin A$ to represent the motion of Boat C. The constants $x_0$ and $y_0$ represent the boats' starting coordinates. Verify that these equations model Boat C's motion accurately. |

The investigation showed you how motion can be affected by another factor, such as wind or current. In this example you will explore what happens when a pilot doesn't compensate for the effect of wind.

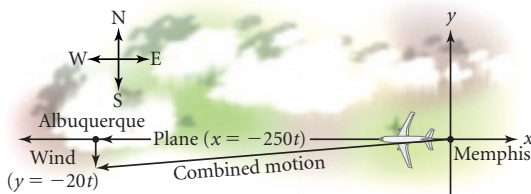
### EXAMPLE A

A pilot heads a plane due west from Memphis, Tennessee, toward Albuquerque, New Mexico. The cities are 1000 mi apart, and the pilot sets the plane's controls to fly at 250 mi/h. However, there is a constant 20 mi/h wind blowing from the north. Where does the plane end up after 4 h?



### ► Solution

Set up a coordinate system with Memphis at the origin.

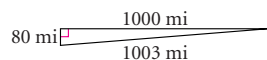


The plane's motion can be described by the equation  $x = -250t$ . Notice that velocity is negative because the plane is flying west. The effect of the wind on the plane's motion can be described by the equation  $y = -20t$ . Why is the wind's velocity negative?

Graph  $x = -250t$  and  $y = -20t$  in an appropriate window with  $0 \leq t \leq 4$ .

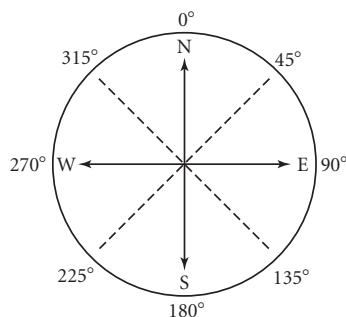
Solving the equation  $-1000 = -250t$ , or tracing the graph, will show that after 4 h the plane has traveled the necessary 1000 mi west, but it is also 80 mi south of Albuquerque, somewhere over the White Sands Missile Range!

The Pythagorean Theorem indicates that the plane has actually traveled  $\sqrt{1000^2 + 80^2}$ , or about 1003 mi, in 4 h.



Because speed is the distance traveled per unit of time, you can find the plane's speed by dividing  $\frac{1003 \text{ mi}}{4 \text{ h}}$  to get the rate of 250.8 mi/h. This tells you that, while the air speed of the plane was 250 mi/h, the ground speed was 250.8 mi/h. The air speed is the plane's flying velocity, and the ground speed is its rate of motion relative to a fixed point on the ground.

The actual angle of motion in Example A is  $\tan^{-1}\left(\frac{80}{1000}\right)$ , or about  $4.57^\circ$ . To determine the plane's bearing, look at the compass rose below. The bearing is always between  $0^\circ$  and  $360^\circ$  measured clockwise from north. This plane intended to travel west ( $270^\circ$ ), but actually traveled south of west. Its bearing was  $270^\circ - 4.57^\circ$ , or  $265.43^\circ$ .



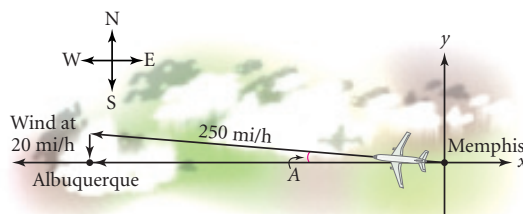
This portable compass and sundial dates from China's Ming Dynasty (1368–1644 C.E.).



### EXAMPLE B

What angle and bearing to the nearest hundredth of a degree should the pilot in Example A set so that the plane actually lands in Albuquerque?

#### ► Solution

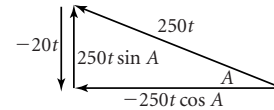


Set up a coordinate system with Memphis at the origin. Because the plane will be blown to the south, it should actually head in a direction slightly to the north.



Sketch the plane's path. The plane flies 250 mi/h, so its distance along this path is  $250t$ . This distance is broken into two separate components. The east-west component is  $x = -250t \cos A$ , and the north-south component is  $y = 250t \sin A$  where  $A$  is the angle toward the north that the pilot must set.

The effect of the wind on the plane's motion can be described by the equation  $y = -20t$ . The sum of the north-south component of the plane's course and the wind's velocity must be 0 if the pilot hopes to land in Albuquerque.



$$250t \sin A + (-20t) = 0$$

The sum of the north-south velocities of the wind and the plane is 0.

$$250t \sin A = 20t$$

Add  $20t$  to both sides.

$$250 \sin A = 20$$

Divide by  $t$ . (Assume  $t \neq 0$ .)

$$\sin A = 0.08$$

Divide both sides by 250, and evaluate.

$$A = \sin^{-1} 0.08$$

Take the inverse sine of both sides.

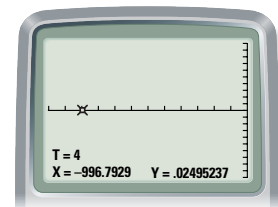
$$A \approx 4.59^\circ$$

Evaluate.

The plane must head out at  $4.59^\circ$  north of west. This is equivalent to a bearing of  $270^\circ + 4.59^\circ$ , or  $274.59^\circ$ .

Graph  $x = -250t \cos 4.59^\circ$  and  $y = 250t \sin 4.59^\circ - 20t$  to verify that the plane moves directly west. As you trace the graph, notice that the value of  $y$  increases slightly. By the time the plane has traveled west 996.79 mi, the plane is slightly north of Albuquerque by 0.02 mi. This means that the angle,  $4.59^\circ$ , is not accurate enough to result in motion directly west. Depending upon the sensitivity of the real-world situation, the pilot may need to increase accuracy by using more decimal places.

Also notice that the plane does not reach Albuquerque within 4 h. Because of the effect of the wind, the plane must travel slightly longer to complete the 1000 mi trip.



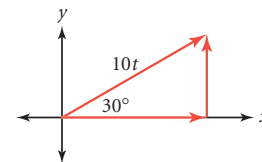
$[-1200, 0, 100, -10, 10, 1]$

The directed distances in this lesson are **vectors**. Vectors are used when you are working with quantities that have both a magnitude (size) and a direction, such as the velocity of the wind, the force of gravity, and the forces you feel when you ride a roller coaster. Working with vectors is a common practice in many fields, including physics, geology, and engineering. The exercises will give you an introduction to this valuable tool.

## EXERCISES

### Practice Your Skills

1. An object is moving at a speed of 10 units per second, at an angle of  $30^\circ$  above the  $x$ -axis, as shown. What parametric equations describe the horizontal and vertical components of this motion?

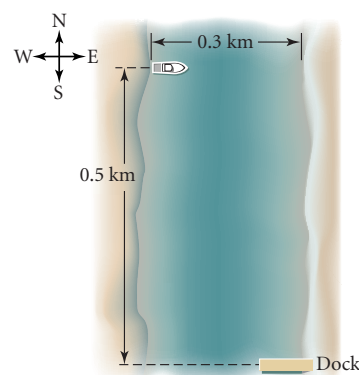


2. Draw a compass rose and vector with magnitude  $v$  to find the bearing of each direction.
  - a.  $14^\circ$  south of east      b.  $14^\circ$  east of south      c.  $14^\circ$  south of west      d.  $14^\circ$  north of west
3. Draw a compass rose and vector with magnitude  $v$  for each bearing. Find the angle made with the  $x$ -axis.
  - a.  $147^\circ$       b.  $204^\circ$       c.  $74^\circ$       d.  $314^\circ$
4. Give the sign of each component vector for the bearings in Exercise 3. For example, for a bearing of  $290^\circ$ ,  $x$  is negative and  $y$  is positive.



## Reason and Apply

5. **APPLICATION** A river is 0.3 km wide and flows south at a rate of 7 km/h. You start your trip on the river's west bank, 0.5 km north of the dock, as shown in the diagram at right.
  - a. If the dock is at the origin,  $(0, 0)$ , what are the coordinates of the boat's starting location?
  - b. Write an equation for  $x$  in terms of  $t$  that models the boat's horizontal position if you aim the boat directly east traveling at 4 km/h.
  - c. Write an equation for  $y$  in terms of  $t$  that models the boat's vertical position, as a result of the flow of the river.
  - d. Enter the parametric equations from 5b and 5c into your calculator, determine a good viewing window and range of  $t$ -values, and make a graph to simulate this situation.
  - e. Determine when and where the boat meets the river's east bank. Does your boat arrive at the dock?
  - f. How far have you traveled?



6. **APPLICATION** A pilot wants to fly from Toledo, Ohio, to Chicago, Illinois, which lies 280 mi directly west. Her plane can fly at 120 mi/h. She ignores the wind and heads directly west. However, there is a 25 mi/h wind blowing from the south.
  - a. Write the equation that describes the effect of the wind.
  - b. Write the equation that describes the plane's contribution to the motion.
  - c. Graph these equations.
  - d. How far off course is the plane after traveling 280 mi west?
  - e. How far has the plane actually traveled?
  - f. What was the plane's ground speed?



7. Fred rows his small boat directly across a river, which is 4 mi wide. There is a 5 mi/h current. When he reaches the opposite shore, Fred finds that he has landed at a point 2 mi downstream.
  - a. Write the equation that describes the effect of the river current.
  - b. If Fred's boat can go  $s$  mi/h, what equation describes his contribution to the motion?

- c. Solve the system of equations from 7a and 7b for  $s$  so that he reaches the correct point 2 mi downstream on the opposite shore.
- d. How far did Fred actually travel?
- e. How long did it take him?
- f. What was Fred's actual speed?
- g. As the boat travels down the river, what angle does it make with the riverbank?

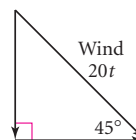


8. **APPLICATION** A plane takes off from Orlando, Florida, heading 975 mi due north toward Cleveland, Ohio. The plane flies at 250 mi/h, and there is a 25 mi/h wind blowing from the west.

- a. Where is the plane after it has traveled 975 mi north?
- b. How far did the plane actually travel?
- c. How fast did the plane actually travel?
- d. At what angle from the east-west axis did the plane actually fly?
- e. What was the bearing?

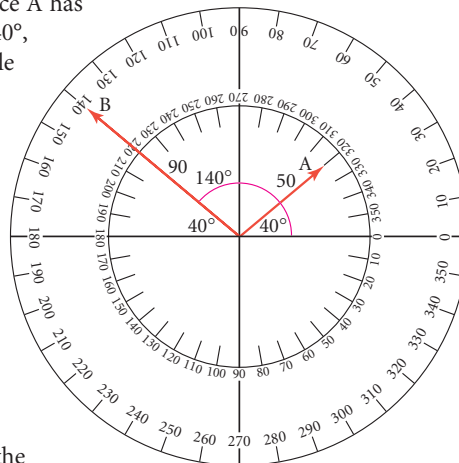


9. **APPLICATION** A plane is headed from Memphis, Tennessee, to Albuquerque, New Mexico, 1000 mi due west. The plane flies at 250 mi/h, and the pilot encounters a 20 mi/h wind blowing from the northwest. (That means the direction of the wind makes a  $45^\circ$  angle below the  $x$ -axis.)
  - a. Write an equation modeling the southward component of the wind.
  - b. Write an equation modeling the eastward component of the wind.
  - c. If the pilot does not compensate for the wind, explain why the final equations for the flight are  $x = -250t + 20t \cos 45^\circ$  and  $y = -20t \sin 45^\circ$ .
  - d. What graphing window and range of  $t$ -values can you use to simulate this flight?
  - e. Solve the equation  $-1000 = -250t + 20t \cos 45^\circ$ . What is the real-world meaning of your answer?
  - f. Use your answer from 9e to find how far south of Albuquerque the plane ended up.



10. **APPLICATION** For the plane in Exercise 9 to land in Albuquerque, it must head a bit north. Let  $A$  represent the measure of the angle north of west.
  - a. For the plane to fly directly west, the northward component of the plane's motion and the wind's southward component must sum to zero. At what angle must the plane fly to head directly to Albuquerque? Write an equation and solve for  $A$ .
  - b. Using the value of  $A$  you found in 10a, write parametric equations to model the plane's flight, and graph them to verify that the plane travels directly west.
  - c. For what bearing should the pilot set his instruments?

- 11. APPLICATION** A plane is flying on a bearing of  $310^\circ$  at a speed of 320 mi/h. The wind is blowing directly from the east at a speed of 32 mi/h.
- Make a compass rose and vector to indicate the plane's motion.
  - Write equations that model the plane's motion without the wind.
  - Make a compass rose and vector to indicate the wind's motion.
  - Write equations that model the wind's motion.
  - What are the resulting equations that model the motion of the plane with the wind?
  - Where is the plane after 5 h?
- 12.** Angelina wants to travel directly across the Wyde River, which is 2 mi wide in this stretch. Her boat can move at a speed of 4 mi/h. The river current flows south at 3 mi/h. At what angle upstream should she aim the boat so that she ends up going straight across?
- 13. APPLICATION** Two forces are pulling on an object. Force A has magnitude 50 newtons (N) and pulls at an angle of  $40^\circ$ , and Force B has magnitude 90 N and pulls at an angle of  $140^\circ$ , as shown. (Note that the lengths of the vectors show their magnitude.)
- Find the  $x$ - and  $y$ -components of Force A.
  - Find the  $x$ - and  $y$ -components of Force B.
  - What is the sum of the  $x$ - and  $y$ -components of the two forces?
  - Using the Pythagorean Theorem, find the magnitude of the resulting force.
  - Find the angle of the resulting force. If your angle is negative, add  $180^\circ$  to get the direction.
  - What additional force will balance Forces A and B and keep the object in equilibrium? (It should be the same magnitude as the sum of Forces A and B, but in the opposite direction.)



### Science CONNECTION

Biomechanics provides an understanding of the internal and external forces acting on the human body during movement. Knowing the role muscles play in generating force and controlling movement is necessary to understanding the limitations of human motion. Information gained from biomechanics helps athletes prepare better for their sports, and sporting goods manufacturers produce better equipment. Research in biomechanics also contributes to better treatment and rehabilitation in case of injury. You can learn more about the study and application of biomechanics with the Internet links at [www.keymath.com/DAA](http://www.keymath.com/DAA).

Yoga is a form of exercise that focuses on breathing, stretching, balance, and meditation. In balancing poses, yoga practitioners must carefully balance the forces that act on the body.





## LESSON

# 8.5

*It is a mathematical fact that the casting of this pebble from my hand alters the centre of gravity of the universe.*

THOMAS CARLYLE

## Projectile Motion

In this lesson you will model motion affected by gravity, building on your earlier work with the quadratic equation  $y = at^2 + v_0t + s_0$ . Recall that  $a$  is based on the acceleration due to gravity, so the equation can also be written as  $y = -\frac{1}{2}gt^2 + v_0t + s_0$ . This equation models the changing vertical height of an object in free fall. Recall that  $t$  represents time,  $v_0$  represents the vertical component of the initial velocity, and  $s_0$  represents the initial height. Some numerical values for the force of gravity,  $g$ , are given below. What are the corresponding values of  $a$ ?

| $g$                | Earth | Moon | Mars |
|--------------------|-------|------|------|
| m/s <sup>2</sup>   | 9.8   | 1.6  | 3.7  |
| ft/s <sup>2</sup>  | 32    | 5.3  | 12   |
| cm/s <sup>2</sup>  | 980   | 162  | 370  |
| in./s <sup>2</sup> | 384   | 64   | 144  |

In Lesson 7.3, you dealt with only the vertical component, the height, of objects in projectile motion. In this lesson you'll model the horizontal component as well, using parametric equations.

### EXAMPLE A

A ball rolls off the end of a table with a horizontal velocity of 1.5 ft/s. The table is 2.75 ft high.

- What equation describes the  $x$ -direction position of the ball?
- Write an equation to model the vertical position of the ball.
- Enter these two equations into your calculator and graph them. Where and when does the ball appear to hit the floor?
- Solve algebraically to find where and when the ball hits the floor.

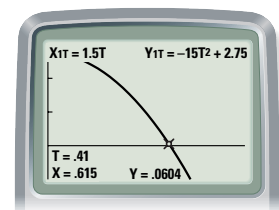


### ► Solution

The vertical component of the ball's motion is affected by the force of gravity, whereas the horizontal component is not.

- The horizontal component of the motion is described by  $x = 1.5t$ . The total distance traveled is the velocity multiplied by the time elapsed.
- An equation for the height is  $y = -\frac{1}{2} \cdot 32 \cdot t^2 + 0 \cdot t + 2.75$ . Here,  $t$  is the time measured in seconds, 32 is the acceleration due to gravity in feet per second squared, the coefficient 0 implies that none of the initial velocity is directed upward or downward, and 2.75 is the initial height in feet.

- c. Graph the equations  $x = 1.5t$  and  $y = -16t^2 + 2.75$  with  $0 \leq t \leq 1$  and a  $t$ -step of 0.01. The ball hits the floor when  $y = 0$ . Trace the graph to approximate when this happens. The graph shows that the ball hits the floor approximately 0.41 s after it leaves the edge of the table. At that time it is 0.62 ft from a point directly below the edge of the table.



[0, 1, 1, -1, 3, 1]

- d. The ball hits the floor at the point where  $y = 0$ . Solve the equation  $0 = -16t^2 + 2.75$ .

$$0 = -16t^2 + 2.75$$

Original equation.

$$16t^2 = 2.75$$

Add  $16t^2$  to both sides.

$$t^2 = \frac{2.75}{16}$$

Divide both sides by 16.

$$t = \pm \sqrt{\frac{2.75}{16}}$$

Take the square root of both sides.

$$t \approx \pm 0.41$$

Evaluate.

Only the positive answer makes sense in this situation, so the ball hits the ground at approximately 0.41 s. To find where this occurs, substitute 0.41 for  $t$  into the parametric equation for  $x$ .

$$x = 1.5t$$

$$x = 1.5(0.41) \approx 0.62$$

The ball hits the ground at a horizontal distance of approximately 0.62 ft from the edge of the table.

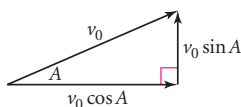
In Example A, the initial motion was only in a horizontal direction. In the previous chapter, you explored motion that was only in a vertical direction, like a free-falling object. Using trigonometric functions, you can parametrically model motion that begins at an angle, using techniques similar to those in the previous lesson.

### Parametric Equations for Projectile Motion

You can model projectile motion parametrically with these equations, where  $x$  is a measure of horizontal position,  $y$  is a measure of vertical position, and  $t$  is a measure of time.

$$x = v_0 t \cos A + x_0$$

$$y = -\frac{1}{2}gt^2 + v_0 t \sin A + y_0$$

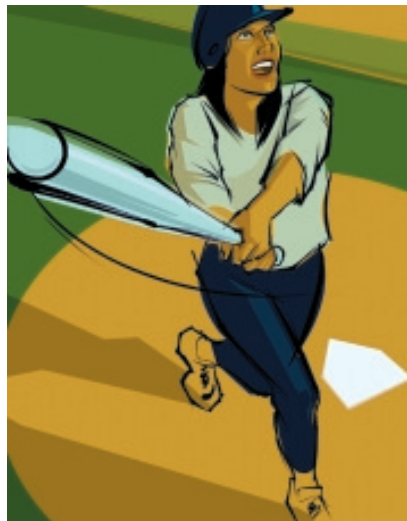


The point  $(x_0, y_0)$  is the initial position at time  $t = 0$ ,  $v_0$  is the velocity at time  $t = 0$ ,  $A$  is the angle of motion from the horizontal, and  $g$  is the acceleration due to gravity.

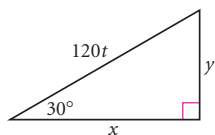


**EXAMPLE B**

Carolina hits a baseball so that it travels at an initial speed of 120 ft/s and at an angle of  $30^\circ$  above the ground. If her bat contacts the ball at a height of 3 ft above the ground, how far does the ball travel horizontally before it hits the ground?

**► Solution**

Draw a picture and write equations for the  $x$ - and  $y$ -components of the motion.



$$\cos 30^\circ = \frac{x}{120t}$$

$$x = 120t \cos 30^\circ$$

$$\sin 30^\circ = \frac{y}{120t}$$

$$y = 120t \sin 30^\circ$$

The horizontal motion is affected only by the initial speed and angle, so the horizontal distance is modeled by  $x = 120t \cos 30^\circ$ .

The vertical motion is also affected by the force of gravity pulling the ball down. In this case the equation that models the vertical distance is  $y = -16t^2 + 120t \sin 30^\circ + 3$ . Compare this equation with the one given in the box on the previous page. Notice how the values from this problem appear in the equation.

To find out when the ball hits the ground, find when the  $y$ -value is 0.

$$-16t^2 + 120t \sin 30^\circ + 3 = 0$$

Original equation.

$$-16t^2 + 120t(0.5) + 3 = 0$$

Evaluate  $\sin 30^\circ$ .

$$-16t^2 + 60t + 3 = 0$$

Multiply.

$$t = \frac{-60 \pm \sqrt{60^2 - 4(-16)(3)}}{2(-16)}$$

Use the quadratic formula. Substitute  $-16$  for  $a$ ,  $60$  for  $b$ , and  $3$  for  $c$ .

$$t \approx -0.049 \text{ or } t \approx 3.799$$

Evaluate.

A negative value for  $t$  doesn't make sense, so use only the positive answer. The ball reaches the ground about 3.8 seconds after being hit.

To determine how far the ball has traveled, substitute this  $t$ -value into the parametric equation for  $x$ :

$$x = 120(3.799) \cos 30^\circ$$

$$x \approx 395$$

The ball will travel about 395 ft horizontally.

In the investigation you will simulate a basketball free throw. This is another instance where you must consider motion at an angle.

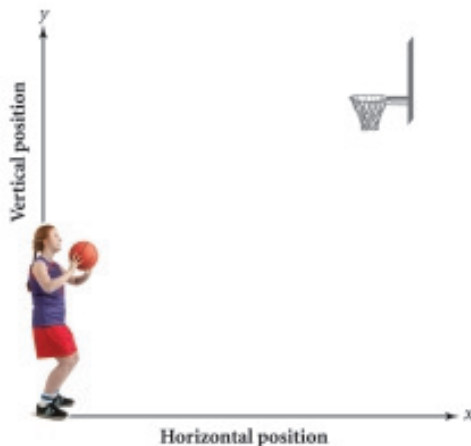


## Investigation

### Basketball Free Throw

(Based on an activity developed by Arne Engebretsen.)

Basketball players decide upon the angle and velocity of the ball necessary to make a basket from various positions on the court. In this investigation you'll explore the relationship between the angle and velocity necessary for a successful free throw shot.



- |        |  |
|--------|--|
| Step 1 | Discuss with your class or get data from your teacher on free throw measurements. You'll need the distance from the floor to the rim of the basket, the horizontal distance from the free throw line to the backboard, the diameter of the basket, and the length of the bracket that fastens the basket to the backboard. |
| Step 2 | Draw a diagram with the measurements from Step 1, and sketch the typical path of a successful free throw.  |
| Step 3 | Using the free throw line as the origin, $(0, 0)$ , what are the coordinates of the front and back rims of the basket? Plot these two data points in a graphing window with $0 \leq x \leq 18.8$ and $0 \leq y \leq 12.4$ .  |
| Step 4 | Create some reasonable data for a free throw shooter. Include the height of the ball at release ( $y_0$ ), the measure of the angle at release ( $A$ ), and the initial velocity of the ball ( $v_0$ ).  |
| Step 5 | Using your data from Step 4, write parametric equations, $x = v_0 t \cos A$ and $y = -16t^2 + v_0 t \sin A + y_0$ , to model a free throw. Use your equations to simulate a free throw on your calculator.   |
| Step 6 | Experiment with different values of $y_0$ , $A$ , and $v_0$ until you can simulate a successful free throw on your graphing calculator.  |
| Step 7 | Is there only one combination of values that will produce a successful free throw? If so, why? If not, generalize patterns in the relationships between the variables.   |

## EXERCISES

### Practice Your Skills

1. A projectile's motion is described by the equations

$$x = -50t \cos 30^\circ + 400$$

$$y = -81t^2 + 50t \sin 30^\circ + 700$$

- a. Is this projectile motion occurring on Earth, the Moon, or Mars? What are the units used in the problem? (*Hint*: Look for the value of  $g$ .)
- b. What is the initial position? Include units in your answer.
- c. At time  $t = 0$ , what direction is the projectile moving? (up, up-right, right, down-right, down, down-left, left, or up-left)
- d. What is the initial velocity? Include units.
2. Find the position at the time given of a projectile in motion described by the equations
- $$x = -50t \cos 30^\circ + 40$$
- $$y = -81t^2 + 50t \sin 30^\circ + 60$$
- a. 0 seconds
- b. 1 second
- c. 2 seconds
- d. 4 seconds
3. A ball rolls off the edge of a 12 m tall cliff at a velocity of 2 m/s.
- a. Write parametric equations to simulate this motion.
- b. What equation can you solve to determine when the ball hits the ground?
- c. When and where does the ball hit the ground?
- d. Describe a graphing window that you can use to model this motion.
4. Consider the scenario in Exercise 3. When and where will the ball hit the ground if the motion occurred
- a. On the Moon?
- b. On Mars?



In dog and disc competitions, the Frisbee is one projectile (although it does face wind resistance) and the dog is another. Each object has a horizontal and vertical component to its motion.

### Reason and Apply

5. These parametric equations model projectile motion of an object:
- $$x = 6t \cos 52^\circ$$
- $$y = -4.9t^2 + 6t \sin 52^\circ + 2$$
- a. Name a graphing window and a range of  $t$ -values that allow you to simulate the motion.
- b. Describe a scenario for this projectile motion. Include a description of every variable and number listed in the parametric equations.

6. A ball rolls off a 3 ft high table and lands at a point 1.8 ft away from the table.
- How long did it take for the ball to hit the floor? Give your answer to the nearest hundredth of a second.
  - How fast was the ball traveling when it left the table? Give your answer to the nearest hundredth, and include units.
7. **APPLICATION** An archer aims at a target 70 m away with diameter 1.22 m. The bull's-eye is 1.3 m above the ground. She holds her bow level at a height of 1.2 m and shoots an arrow with an initial velocity of 83 m/s.
- What equations model this motion?
  - Will she hit the target? If not, by how much will she miss it?
  - At what range of angles could the archer hold her bow to hit somewhere on the target?
  - What initial velocity is necessary in order to hit the target if she holds her bow level?

### Sports CONNECTION

Archery is one of the oldest sports that is still practiced today. Modern archery became an Olympic event in 1972. Olympic archers aim at a target 1.22 m in diameter, 70 m away—the bull's-eye is just 12.2 cm in diameter. From where the archers stand, the target looks about the same size as the head of a thumbtack held at arm's length. Archery is used as a source of rehabilitation and recreation worldwide.



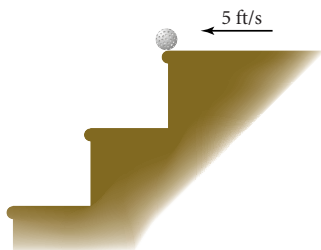
Paola Fantato of Italy won the gold medal in archery in the Women's Individual Sitting Contest at the 2000 Paralympic Games in Sydney, Australia.

- By how much does the ball in Example B clear a 10 ft fence that is 365 ft away if the wind is blowing directly from the fence toward Carolina at 8 mi/h?
- Gonzo, the human cannonball, is fired out of a cannon 10 ft above the ground at a speed of 40 ft/s. The cannon is tilted at an angle of  $60^\circ$ . His net hangs 5 ft above the ground. Where does his net need to be positioned so that he will land safely?



Pictured here in Berlin, Germany, a spring-loaded cannon propelled the "Human Cannonball" of the 1920s, Paul Leinert.

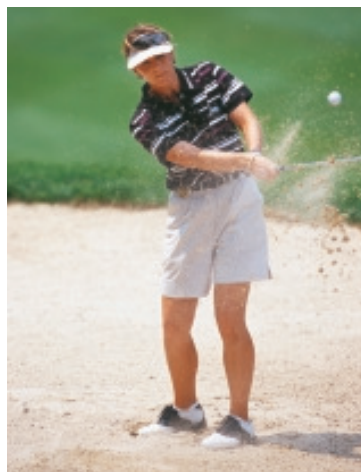
10. A golf ball rolls off the top step of a flight of 14 stairs with a horizontal velocity of 5 ft/s. The stairs are each 8 in. high and 8 in. wide. On which step does the ball first bounce?



### Sports

#### CONNECTION

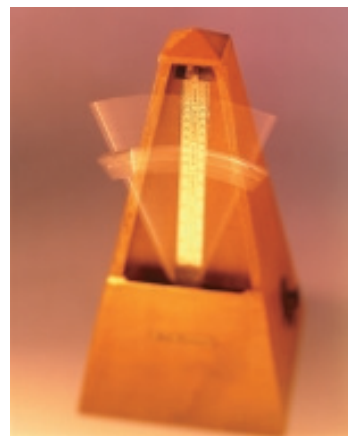
Golfers have a range of golf clubs used for hitting distance shots, including “irons” and “woods.” Golf clubs have faces with a variety of angles. A club with a lesser angle, such as a No. 1 wood, is referred to as having less “loft”—the loft is the angle at which the ball should leave the ground. A low loft will cause the ball to leave the ground at a low angle and travel for more distance. A club with a high loft, such as a No. 7 wood, will cause a ball to travel higher and a shorter distance.



Swedish-born Annika Sorenstam (b 1970) is a member of the Ladies Professional Golf Association.

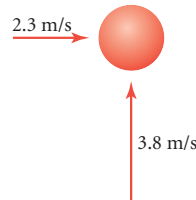
11. A golfer swings a 7-iron golf club with a loft of  $38^\circ$  and an initial velocity of 122 ft/s on level ground.
- Write parametric equations to simulate this golf shot.
  - How far away does the ball first hit ground?
  - How far away would the ball land if a golfer chose a 9-iron golf club with a loft of  $46^\circ$  and an initial velocity of 110 ft/s?
12. The tip of a metronome travels on a path modeled by the parametric equations  $x = 0.7 \sin t$  and  $y = \sqrt{1 - (0.7 \sin t)^2}$ .
- Sketch the graph when  $0^\circ \leq t \leq 360^\circ$ . Describe the motion.
  - Eliminate the parameter,  $t$ , and write a single equation using only  $x$  and  $y$ . Sketch this graph.
  - Compare the two graphs.
  - Rewrite the parametric equations so that the graph is reflected across the  $x$ -axis.
  - Sketch the graph when  $0^\circ \leq t \leq 720^\circ$ . What does changing  $360^\circ$  to  $720^\circ$  do?

A metronome is an instrument that marks exact time for musicians. It can be set to produce a regular beat at a fast or slow pace.



## Review

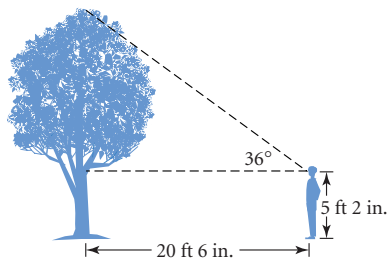
- 13.** Two forces act simultaneously on a ball positioned at  $(4, 3)$ . The first force imparts a velocity of  $2.3 \text{ m/s}$  to the east, and the second force imparts a velocity of  $3.8 \text{ m/s}$  to the north.



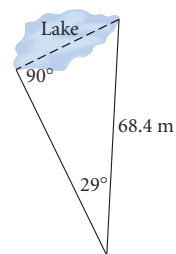
- Enter parametric equations into your calculator to simulate the resulting movement of the ball. Sketch your graph, and give your equations and graphing window.
- What is the ball's velocity? Give the magnitude and direction.

- 14.** Find the height, width, area, and distance specified.

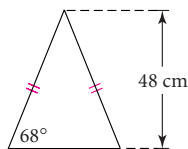
- a.** total height of the tree



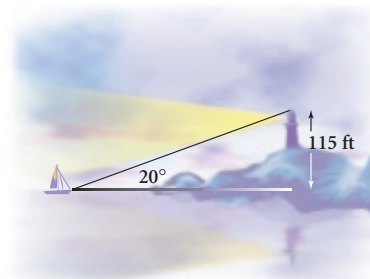
- b.** width of the lake



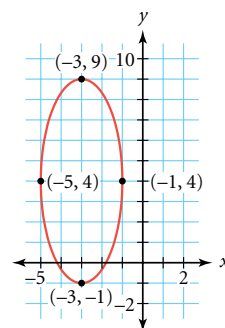
- c.** area of the triangle



- d.** distance from the boat to the lighthouse



- Find a polynomial equation of least degree with integer coefficients that has roots  $-3$  and  $(\frac{1}{2} - \sqrt{3})$ .
- Give parametric equations and a single equation using only  $x$  and  $y$  for the ellipse at right.

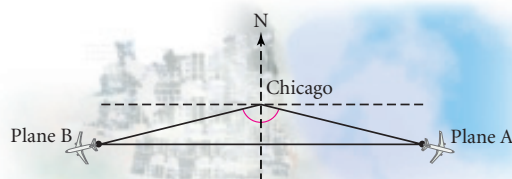


# LESSON

# 8.6

# The Law of Sines

Two airplanes pass over Chicago at the same time. Plane A is cruising at 400 mi/h on a bearing of  $105^\circ$ , and Plane B is cruising at 450 mi/h on a bearing of  $260^\circ$ . How far apart will they be after 2 hours?



The problem above is a familiar scenario, but the triangle formed by the planes' paths is not a right triangle. In the next two lessons, you will discover useful relationships involving the sides and angles of nonright, or **oblique**, triangles and apply those relationships to situations like the distance problem presented above.



## Investigation Oblique Triangles

### You will need

- a ruler
- a protractor

In this investigation you'll explore a special relationship between the sines of the angle measures of an oblique triangle and the lengths of the sides.

- Step 1** Have each group member draw a different acute triangle  $ABC$ . Label the length of the side opposite  $\angle A$  as  $a$ , the length of the side opposite  $\angle B$  as  $b$ , and the length of the side opposite  $\angle C$  as  $c$ . Draw the altitude from  $\angle A$  to  $\overline{BC}$ . Label the height  $h$ .
- Step 2** The altitude divides the original triangle into two right triangles, one containing  $\angle B$  and the other containing  $\angle C$ . Use your knowledge of right triangle trigonometry to write an expression involving  $\sin B$  and  $h$ , and an expression with  $\sin C$  and  $h$ . Combine the two expressions by eliminating  $h$ . Write your new expression as a proportion in the form
- $$\frac{\sin B}{?} = \frac{\sin C}{?}$$
- Step 3** Now draw the altitude from  $\angle B$  to  $\overline{AC}$  and label the height  $j$ . Repeat Step 2 using expressions involving  $j$ ,  $\sin C$ , and  $\sin A$ . What proportion do you get when you eliminate  $j$ ?
- Step 4** Compare the proportions that you wrote in Steps 2 and 3. Use the transitive property of equality to combine them into an extended proportion:
- $$\frac{?}{?} = \frac{?}{?} = \frac{?}{?}$$



- Step 5 | Share your results with the members of your group. Did everyone get the same proportion in Step 4?

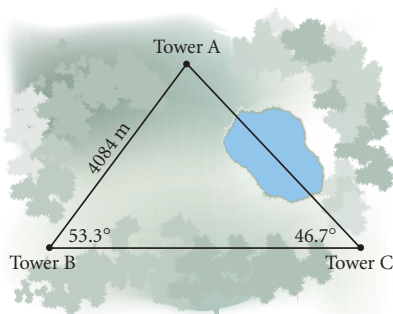
Sine, cosine, and tangent are defined for all real angle measures. (See the exploration on page 447 to learn how these definitions are extended beyond right triangles.) Therefore, you can find the sine of obtuse angles as well as acute angles and right angles. Does your work from Steps 1–5 hold true for obtuse triangles as well?

- Step 6 | Have each group member draw a different obtuse triangle. Measure the angles and the sides of your triangle. Substitute the measurements and evaluate to verify that the proportion from Step 4 holds true for your obtuse triangles as well.

The relationships that you discovered in the investigation allow you to solve many problems involving oblique triangles.

### EXAMPLE A

Towers A, B, and C are located in a national forest. From Tower B, the angle between Towers A and C is  $53.3^\circ$ , and from Tower C the angle between Towers A and B is  $46.7^\circ$ . The distance between Towers A and B is 4084 m. A lake between Towers A and C makes it difficult to measure the distance between them directly. What is the distance between Towers A and C?



This forest fire observation tower is located in Great Smoky Mountains National Park, Tennessee.

► **Solution**

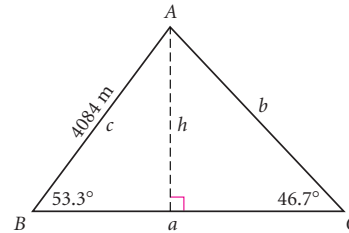
First, convince yourself that  $\triangle ABC$  is not a right triangle. The sum of the measures of the angles in a triangle is  $180^\circ$ , so  $\angle A$  measures  $80^\circ$ .

On your paper, sketch and label a diagram. Include the altitude from  $\angle A$  to  $\overline{BC}$ . As in the investigation, two right triangles are formed. Set up sine ratios for each.

$$\sin 53.3^\circ = \frac{h}{4084} \quad \text{or} \quad h = 4084 \sin 53.3^\circ$$

and

$$\sin 46.7^\circ = \frac{h}{b} \quad \text{or} \quad h = b \sin 46.7^\circ$$



Substituting for  $h$  gives  $4084 \sin 53.3^\circ = b \sin 46.7^\circ$ . Solving for  $b$  gives

$$b = \frac{4084 \sin 53.3^\circ}{\sin 46.7^\circ}$$

$$b \approx 4499$$

The distance between Towers A and C is approximately 4499 m.

In Example A, notice that the height,  $h$ , was eliminated from the final calculations. You did not need a measurement for  $h$ ! The equation  $4084 \sin 53.3^\circ = b \sin 46.7^\circ$  can also be written as  $\frac{\sin 53.3^\circ}{b} = \frac{\sin 46.7^\circ}{4084}$ , or  $\frac{\sin B}{b} = \frac{\sin C}{c}$ , which is the relationship you found in the investigation.

For all three angles of a triangle the ratio of the sine of an angle to the length of the opposite side is constant. This relationship is called the **Law of Sines**.

**Law of Sines**

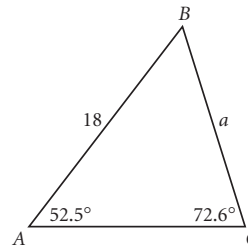
For any triangle with angles  $A$ ,  $B$ , and  $C$ , and sides of lengths  $a$ ,  $b$ , and  $c$  ( $a$  is opposite  $\angle A$ ,  $b$  is opposite  $\angle B$ , and  $c$  is opposite  $\angle C$ ),

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

You use the Law of Sines to find missing parts of triangles when you know the measures of two angles and the length of one side of a triangle.

**EXAMPLE B**

Find the length of  $\overline{BC}$ .



► **Solution**

Use the Law of Sines.

$$\begin{aligned}\frac{\sin A}{a} &= \frac{\sin C}{c} \\ \frac{\sin 52.5^\circ}{a} &= \frac{\sin 72.6^\circ}{18} \\ 18 \sin 52.5^\circ &= a \sin 72.6^\circ \\ \frac{18 \sin 52.5^\circ}{\sin 72.6^\circ} &= a \\ a &\approx 15\end{aligned}$$

Select the proportion for the given angles and sides.

Substitute the angle measures and the known side length.

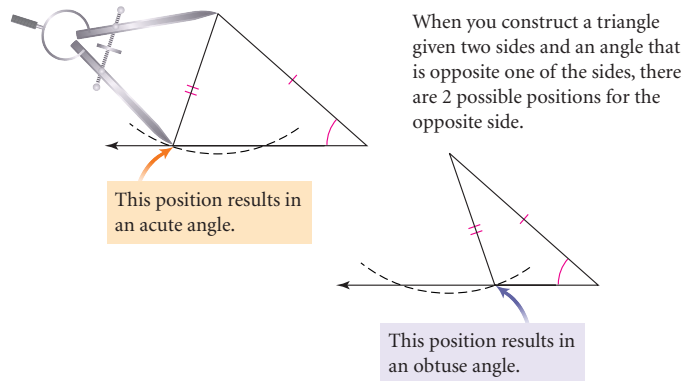
Multiply both sides by  $18a$  and reduce.

Divide both sides by  $\sin 72.6^\circ$  and reduce the right side.

Evaluate.

The length of  $\overline{BC}$  is approximately 15 units.

You may also use the Law of Sines when you know two side lengths and the measure of the angle opposite one of the sides. However, in this case you may find more than one possible solution. This is because two different angles—one acute and one obtuse—may share the same value of sine. Look at this diagram to see how this works.

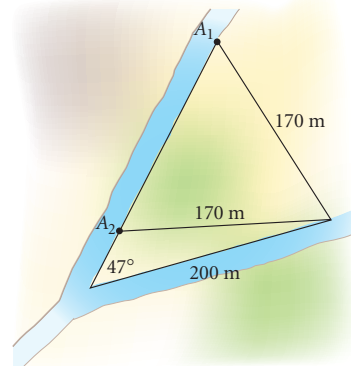


**EXAMPLE C**

Tara and Yacin find a map that they think will lead to buried treasure. The map instructs them to start at the  $47^\circ$  fork in the river. They need to follow the line along the southern branch for 200 m, then walk to a point on the northern branch that's 170 m away. Where along the northern branch should they dig for the treasure?

► **Solution**

As the map shows, there are two possible locations to dig. Consider the angles formed by the 170 m segment and the line along the northern branch as  $\angle A$ . Use the Law of Sines to find one possible measure of  $\angle A$ .



$$\frac{\sin 47^\circ}{170} = \frac{\sin A}{200}$$

Set up a proportion using opposite sides and angles.

$$\sin A = \frac{200 \sin 47^\circ}{170}$$

Solve for  $\sin A$ .

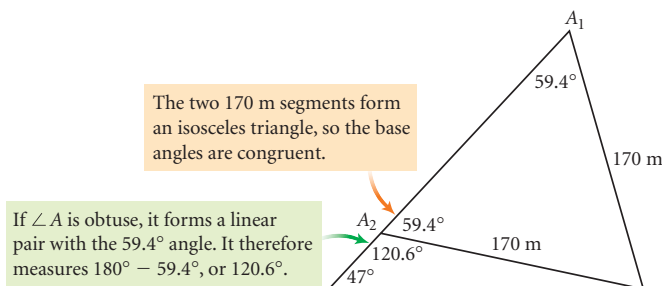
$$A = \sin^{-1}\left(\frac{200 \sin 47^\circ}{170}\right)$$

Take the inverse sine of both sides.

$$A \approx 59.4^\circ$$

Evaluate.

If  $\angle A$  is acute, it measures approximately  $59.4^\circ$ . The other possibility for  $\angle A$  is the obtuse supplement of  $59.4^\circ$ , or  $120.6^\circ$ . You can verify this with geometry as shown in the diagram below. Use your calculator to check that  $\sin 59.4^\circ$  is equivalent to  $\sin 120.6^\circ$  and that both angle measures satisfy the Law of Sines equation  $\frac{\sin 47^\circ}{170} = \frac{\sin A}{200}$ .



In order to find the distance along the northern branch, you need the measure of the third angle in the triangle. Use the known angle measure,  $47^\circ$ , and the approximations for the measure of  $\angle A$ .

$$180^\circ - (47^\circ + 59.4^\circ) \approx 73.6^\circ \quad \text{or} \quad 180^\circ - (47^\circ + 120.6^\circ) \approx 12.4^\circ$$

Use the Law of Sines to find the distance along the northern branch,  $x$ .

$$\frac{\sin 73.6^\circ}{x} = \frac{\sin 47^\circ}{170} \quad \text{or} \quad \frac{\sin 12.4^\circ}{x} = \frac{\sin 47^\circ}{170}$$

$$x = \frac{170 \sin 73.6^\circ}{\sin 47^\circ} \quad \text{or} \quad x = \frac{170 \sin 12.4^\circ}{\sin 47^\circ}$$

$$x \approx 223 \quad \text{or} \quad x \approx 50$$

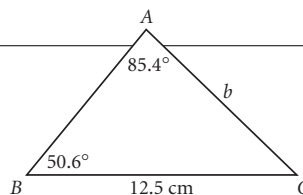
Tara and Yacin should dig two holes along the northern branch, one 50 m and the other 223 m from the fork of the river.

A situation like that in Example C, where more than one possible solution exists, is called an **ambiguous case**. You can't tell which of the possibilities is correct unless there is more information in the problem, such as knowing whether the triangle is acute or obtuse. In general, you should report both solutions in cases like this.

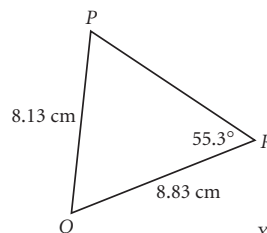
## EXERCISES

### Practice Your Skills

- Find the length of side  $\overline{AC}$ .

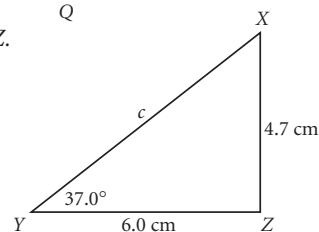


2. Assume  $\triangle PQR$  is an acute triangle. Find the measure of  $\angle P$ .



3. In  $\triangle XYZ$ , at right,  $\angle Z$  is obtuse. Find the measures of  $\angle X$  and  $\angle Z$ .

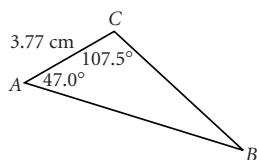
4. Find the length of  $\overline{XY}$  in Exercise 3.



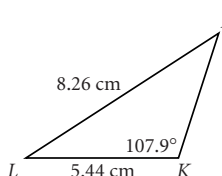
## Reason and Apply

5. Find all of the unknown angle measures and side lengths.

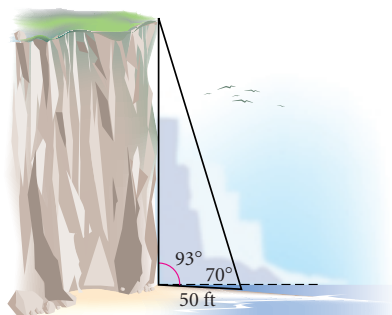
a.



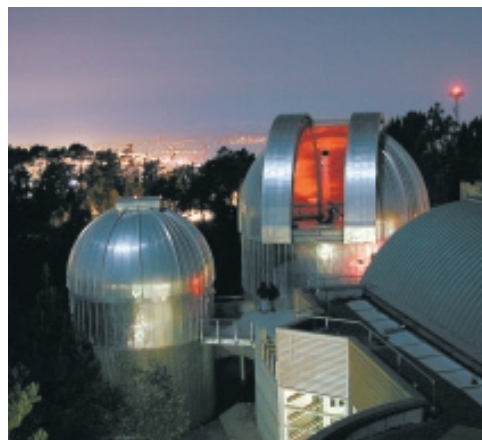
b.



6. The Daredevil Cliffs rise vertically from the beach. The beach slopes gently down to the water at an angle of  $3^\circ$  from the horizontal. Scott lies at the water's edge, 50 ft from the base of the cliff, and determines that his line of vision to the top of the cliff makes a  $70^\circ$  angle with the line to the beach. How high is the cliff?

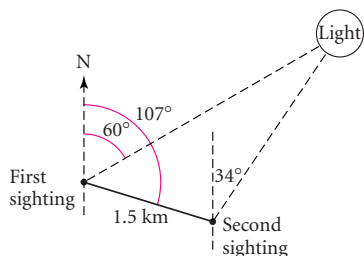


7. In an isosceles triangle, one of the base angles measures  $42^\circ$ . The length of each leg is 8.2 cm.
- Find the length of the base.
  - Even though you are given one angle and two sides not including the angle, this is not an ambiguous case. Why not?
8. **APPLICATION** Venus is 67 million mi from the Sun. Earth is 93 million mi from the Sun. Gayle measures the angle between the Sun and Venus as  $14^\circ$ . At that moment in time, how far is Venus from Earth?



Chabot Observatory in Oakland, California

- 9. APPLICATION** The SS *Minnow* is lost at sea in a deep fog. Moving on a bearing of  $107^\circ$ , the skipper sees a light at a bearing of  $60^\circ$ . The same light reappears through the fog after the skipper has sailed 1.5 km on his initial course. The second sighting of the light is at a bearing of  $34^\circ$ . How far is the boat from the source of the light at the time of the second sighting?

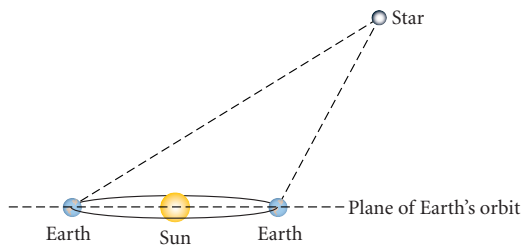


### History CONNECTION

Lighthouses project light through darkness or poor weather to help guide ships ashore. The first lighthouses were actually bonfires—one was even mentioned in the *Iliad*, a Greek epic poem by Homer written sometime before 700 B.C.E. Today, most lighthouses hold powerful electric lights that flash automatically. Every lighthouse has a distinct sequence of flashes that allows ship captains to identify the nearby harbor.



- 10. APPLICATION** One way to calculate the distance between Earth and a nearby star is to measure the angle between the star and the ecliptic (the plane of Earth's orbit) at 6-month intervals. A star is measured at a  $42.13204^\circ$  angle. Six months later, the angle is  $42.13226^\circ$ . The diameter of Earth's orbit is  $3.13 \times 10^{-5}$  light-years. What is the distance to the star at the time of each reading? Use this diagram to help you solve this problem.



### Science CONNECTION

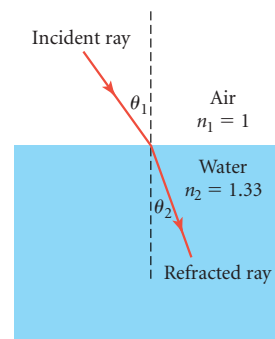
A light-year is the distance that light can travel in one year. Light moves at a velocity of about 300,000 km/s, so one light-year is equal to about 9,461,000,000,000 km. Distances in space are so large that it's difficult to express them with relatively small units such as kilometers. For example, the distance to the next nearest big galaxy is  $21 \times 10^{18}$  km! Another unit of distance used by astronomers to measure distances within our solar system is the astronomical unit (AU). One AU is the average distance between Earth and the Sun, about 150 million km. Pluto averages about 40 AU from the Sun.

- 11. APPLICATION** When light travels from one transparent medium into another, the rays bend, or refract. Snell's Law of Refraction states that

$$\frac{\sin \theta_1}{n_2} = \frac{\sin \theta_2}{n_1}$$

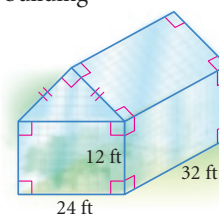
where  $\theta_1$  is the angle of incidence,  $\theta_2$  is the angle of refraction, and  $n_1$  and  $n_2$  are the indexes of refraction for the two mediums, as shown in the diagram.

- Find the angle of refraction in water if the angle of incidence from air is  $60^\circ$ .
- If the angle of refraction from air to water is  $45^\circ$ , at what angle did the ray enter the water?
- If the angle of incidence is  $0^\circ$ , what is the angle of refraction?



## Review

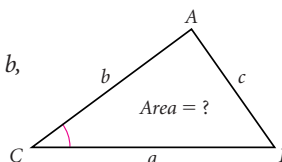
- Use the quadratic formula to solve each equation.
  - $2x^2 - 8x + 5 = 0$
  - $3x^2 + 4x - 2 = 7$
- Draw a compass rose and show each vector. Find the  $x$ - and  $y$ -components of each, and show the angle with the  $x$ -axis used to find the components. Indicate the direction of the components with the proper sign.
  - 12 units on a bearing of  $168^\circ$
  - 16 units on a bearing of  $221^\circ$
- Give the magnitude and bearing for each vector.
  - $x$ -component:  $-9.1$  units  
 $y$ -component:  $4.1$  units
  - $x$ -component:  $16.6$  units  
 $y$ -component:  $14.4$  units
- The value of a building depreciates at a rate of  $6\%$  per year. When new, the building is worth  $\$36,500$ .
  - How much is the building worth after 5 years 3 months?
  - To the nearest month, when will the building be worth less than  $\$10,000$ ?
- Find the volume of the greenhouse at right. Round your answer to the nearest square foot.



## IMPROVING YOUR GEOMETRY SKILLS

### A New Area Formula

You are given the lengths of two sides of a triangle,  $a$  and  $b$ , and the measure of the angle between them,  $C$ . How can you find the area of the triangle using only these three measurements?



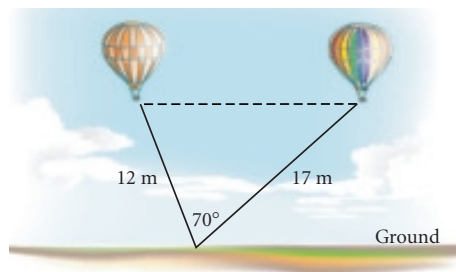


# The Law of Cosines

The Law of Sines enabled you to find the lengths of sides of a triangle or the measures of the angles in certain situations. To use the Law of Sines, you needed to know the measures of two angles and the length of any side, or the lengths of two sides and the measure of the angle opposite one of the sides. What if you know a different combination of sides and angles?

## EXAMPLE A

Two hot-air balloons approach a landing field. One is 12 m from the landing point and the other is 17 m from the landing point. The angle between the balloons is  $70^\circ$ . How far apart are the two balloons?

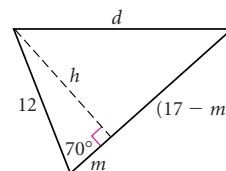


## ► Solution

In this case the Law of Sines does not help. You know only the included angle, or the angle between the two sides. If you try to set up an equation using the Law of Sines, you will always have more than one variable. So you must try something else.

Sketch one altitude to form two right triangles, so that one of the right triangles contains the  $70^\circ$  angle.

If you use the altitude from the balloon on the left, the 17 m side is split into two parts. Label one part  $m$  and the other part  $17 - m$ . Label the height  $h$ , and label the distance between the balloons  $d$ . You can now write two equations using the Pythagorean Theorem.



$$m^2 + h^2 = 12^2 \quad \text{and} \quad (17 - m)^2 + h^2 = d^2$$

You can multiply and expand the binomial in the second equation.

$$289 - 34m + m^2 + h^2 = d^2$$

Now you can rearrange the first equation and use substitution to solve the system of equations.

$$h^2 = 144 - m^2$$

$$289 - 34m + m^2 + (144 - m^2) = d^2$$

$$289 + 144 - 34m = d^2$$

Solve  $m^2 + h^2 = 144$  for  $h^2$ .

Substitute  $144 - m^2$  for  $h^2$  into the second equation.

Combine like terms.

Use the right triangle that contains the  $70^\circ$  angle to write

$$\cos 70^\circ = \frac{m}{12}$$

Solve this equation for  $m$  to get  $m = 12 \cos 70^\circ$ , and substitute this value for  $m$  into the equation for  $d^2$ .

$$289 + 144 - 34(12 \cos 70^\circ) = d^2$$

Substitute  $12 \cos 70^\circ$  for  $m$ .

$$\sqrt{289 + 144 - 34(12 \cos 70^\circ)} = d$$

Take the square root of both sides.

You need only the positive root, because  $d$  represents a distance.

$$d \approx 17.1$$

Evaluate.

The distance between the two balloons is approximately 17.1 m.

You can repeat the procedure that is used in Example A any time you know the lengths of two sides of a triangle and the measure of the included angle, and when you need to find the length of the third side. Notice that you could also write the equation for  $d^2$  as

$$d^2 = 17^2 + 12^2 - 2(17)(12) \cos 70^\circ$$

which looks similar to the Pythagorean Theorem with an extra term that is twice the product of the length of the sides and the cosine of the angle between them. This modified Pythagorean relationship is called the **Law of Cosines**.

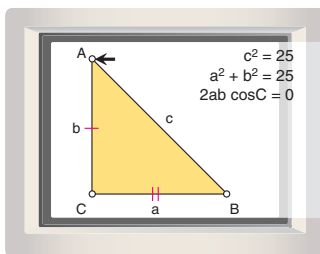
### Law of Cosines

For any triangle with angles  $A$ ,  $B$ , and  $C$  and sides of lengths  $a$ ,  $b$ , and  $c$  ( $a$  is opposite  $\angle A$ ,  $b$  is opposite  $\angle B$ , and  $c$  is opposite  $\angle C$ ),

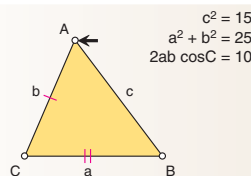
$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

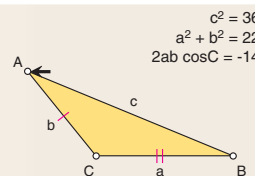
$$c^2 = a^2 + b^2 - 2ab \cos C$$



In this right triangle,  $c^2$  is equivalent to  $a^2 + b^2$ .  
 $c^2 = a^2 + b^2$



In this acute triangle,  $c^2$  is less than  $a^2 + b^2$ . The difference is  $2ab \cos C$ .  
 $c^2 = a^2 + b^2 - 2ab \cos C$



In this obtuse triangle,  $c^2$  is more than  $a^2 + b^2$ . Again, the difference is  $2ab \cos C$ .  
 $c^2 = a^2 + b^2 - 2ab \cos C$

In the investigation you'll apply the Law of Cosines to a real situation.



## Investigation Around the Corner

### You will need

- metersticks or a tape measure
- a protractor

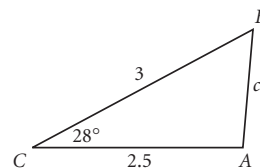
Position three members of your group so that two people are on opposite sides of a wall and the third person can see both of them. Use trigonometry to find the distance between the two persons on opposite sides of the wall. Sketch a diagram of the situation, show the measurements you make, and show your calculations.



You can use the Law of Cosines alone or in combination with other triangle properties.

### EXAMPLE B

Find the unknown angle measures and side lengths.



### ► Solution

First, use the Law of Cosines to find the length of  $\overline{AB}$ .

$$c^2 = a^2 + b^2 - 2ab \cos C$$

The Law of Cosines for finding  $c$  when  $a$ ,  $b$ , and  $C$  are known.

$$c^2 = 3^2 + 2.5^2 - 2(3)(2.5) \cos 28^\circ$$

Substitute 3 for  $a$ , 2.5 for  $b$ , and  $28^\circ$  for  $C$ .

$$c^2 = 9 + 6.25 - 15 \cos 28^\circ$$

Multiply.

$$c = \sqrt{9 + 6.25 - 15 \cos 28^\circ}$$

Take the positive square root of both sides.

$$c \approx 1.42$$

Evaluate.

The length of  $\overline{AB}$  is approximately 1.42 units.

Now use the Law of Cosines to find the measure of  $\angle A$ .

|   |  |
|---|--|
| $a^2 = b^2 + c^2 - 2bc \cos A$                        | The Law of Cosines for finding $A$ when $a$ , $b$ , and $c$ are known. |
| $3^2 \approx 2.5^2 + 1.42^2 - 2(2.5)(1.42) \cos A$    | Substitute values for $a$ , $b$ , and $c$ .                            |
| $9 \approx 8.2664 - 7.1 \cos A$                       | Multiply.  |
| $0.7336 \approx -7.1 \cos A$                          | Subtract 8.2664 from both sides.                                       |
| $\cos A \approx \frac{0.7336}{-7.1}$                  | Divide by $-7.1$ .   |
| $A \approx \cos^{-1}\left(\frac{0.7336}{-7.1}\right)$ | Take the inverse cosine of both sides.                                 |
| $A \approx 96^\circ$                                  | Evaluate.  |

Angle  $A$  measures approximately  $96^\circ$ .

To find the measure of the last angle, use the fact that the measures of the three angles in any triangle sum to  $180^\circ$ . The measure of  $\angle B$  is approximately  $180^\circ - 28^\circ - 96^\circ$ , or  $56^\circ$ .

During a calculation, it is best to use the entire previous answer for the next calculation. Rounding before the last step can reduce the accuracy of your answer. In Example B, you could find the measure of  $\angle A$  with more accuracy by using  $\sqrt{9 + 6.25 - 15 \cos 28^\circ}$  for  $c$  instead of the approximation of 1.42. In all cases, you need to verify that the answers you get make sense in the context of the problem or in a sketch of the triangle.

In deciding whether to use the Law of Sines or the Law of Cosines, consider the triangle parts whose measurements you know and their relationships to each other.

#### Law of Sines

Side-Angle-Angle

Side-Side-Angle (ambiguous case)

#### Law of Cosines

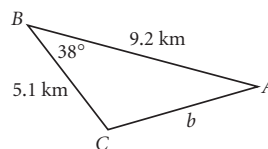
Side-Angle-Side

Side-Side-Side

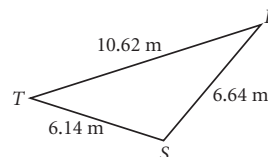
## EXERCISES

### Practice Your Skills

- Find the length of  $\overline{AC}$ .



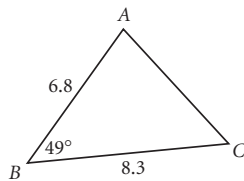
- Find the measure of  $\angle T$ .



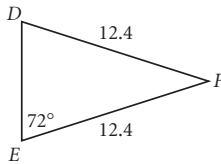
- Solve for  $A$  and  $b$ . (Assume  $b$  is positive.)
  - $16 = 25 + 36 - 2(5)(6) \cos A$
  - $49 = b^2 + 9 - 2(3)(b) \cos 60^\circ$

4. Find all of the unknown angle measures and side lengths.

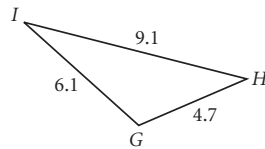
a.



b.



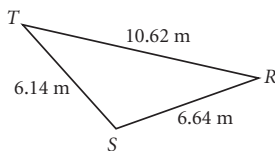
c.



## Reason and Apply

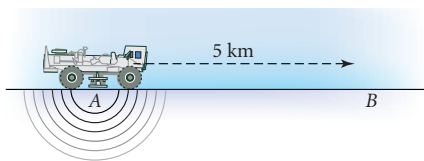
5. Two airplanes pass over Chicago, Illinois, at the same time. One is cruising at 400 mi/h on a bearing of  $105^\circ$ , and the other is cruising at 450 mi/h on a bearing of  $260^\circ$ . How far apart will they be after 2 h?

6. Find the measure of  $\angle S$ .



Pilots use the control panels of this passenger aircraft to follow air traffic control commands, to adjust for wind currents, and to avoid the airspace of other planes.

7. **APPLICATION** Seismic exploration identifies underground phenomena, such as caves, oil pockets, and rock layers, by transmitting sound into the earth and timing the echo of the vibration. From a sounding at point  $A$ , a “thumper” truck locates an underground chamber 7 km away. Moving to point  $B$ , 5 km from point  $A$ , the truck takes a second sounding and finds the chamber is 3 km away from that point. Assume that the underground chamber lies in the same vertical plane as  $A$  and  $B$ . What more can you say about the location of the underground chamber?



## Science

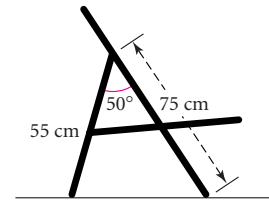
### CONNECTION

Sources of oil are often found via seismic exploration. At one time, seismologists used explosives to generate shock waves, but now they use “thumper” trucks that send controlled vibrations through the ground. Because seismic exploration may be disruptive to a particular environment, oil explorers sometimes use gravitational or magnetic exploration to determine the composition of rock formations based upon Earth’s natural gravity or magnetic fields.

8. A folding chair's legs meet to form a  $50^\circ$  angle. The rear leg is 55 cm long and attaches to the front leg at a point 75 cm from the front leg's foot. How far apart are the legs at the floor?



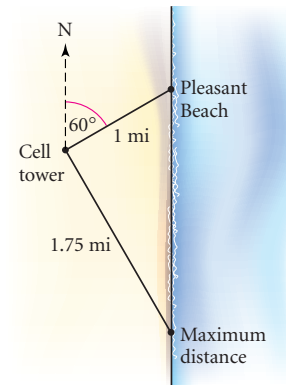
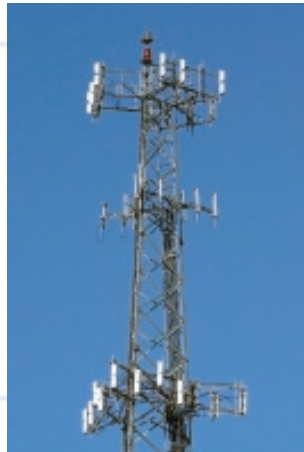
American designer Charles Eames (1907–1978) used mathematics to design this chair, which is suitable for sitting or sprawling.



9. **APPLICATION** The Hear Me Now Phone Company plans to build a cell tower to serve the needs of Pleasant Beach and the beachfront. It decides to locate the cell tower so that Pleasant Beach is 1 mi away at a bearing of  $60^\circ$  from the tower. The range of the signal from the cell tower is 1.75 mi. The beachfront runs north to south. How far south of Pleasant Beach will customers be able to use their cell phones?

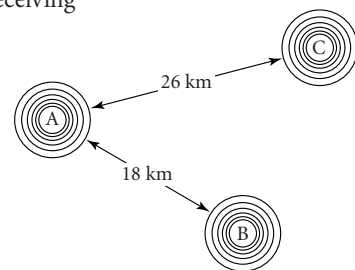
### Technology CONNECTION

The placement of cell towers is crucial to providing a variety of cellular services. Mathematical models are used to analyze possible sites. Cell towers are located by individual companies based on their own business plan for the market they serve. For greatest cost efficiency, cellular companies consider the physical design, topography, population density, environmental impact, engineering, and aesthetics of their towers.



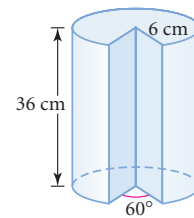
This cell phone tower is located in Jonesboro, Georgia.

10. **APPLICATION** Triangulation is used to locate airplanes, boats, or vehicles that transmit radio signals. The distances to the vehicle are found and the directions calculated by measuring the strength of the signal at three fixed receiving locations. Receiver B is 18 km from Receiver A at a bearing of  $122^\circ$ . Receiver C is 26 km from Receiver A at a bearing of  $80^\circ$ . The signal from a source vehicle to Receiver A indicates a distance of 15 km. The distance from the source vehicle to Receiver B is 8 km and to Receiver C is 25 km. What is the bearing from Receiver A to the source vehicle? (*Hint: You might try making a scale drawing of this situation, and use your compass to get a general idea of the location of the source before starting the calculations.*)

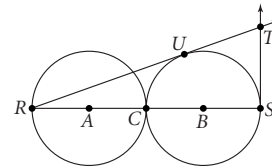


## Review

11. A ship's captain sees a lighthouse at a bearing of  $105^\circ$ . Then the captain sails 8.8 nautical miles on a bearing of  $174^\circ$ . The lighthouse is now at a bearing of  $52^\circ$ . How far is the ship from the lighthouse?
12. Here are the batting averages of the National League's Most Valuable Players from 1980 to 2000. (*The New York Times Almanac 2002*)
- {.286, .316, .281, .302, .314, .353, .290, .287, .290, .291, .301, .319, .311, .336, .368, .319, .326, .366, .308, .319, .334}
- Make a box plot of these data.
  - Give the five-number summary.
  - Find the range and interquartile range.
  - Find the mean and standard deviation.
13. Find the total surface area of the figure at right. Round your answer to the nearest square centimeter.



14. Congruent circles  $A$  and  $B$  are tangent at point  $C$ . The radius of each circle is 3 units. Rays  $RU$  and  $ST$  are tangent at  $U$  and  $S$ , respectively, and intersect at  $T$ . Find  $ST$ .



## Project

### CATAPULT

If you launch a ball at a particular angle and initial velocity, you can determine how far it will travel. Is there another angle at which a ball can be launched, with the same initial velocity, that will cause the ball to travel exactly the same horizontal distance? What angle(s) will cause the ball to travel the farthest distance possible?

Your project should include

- ▶ A conjecture about what kinds of angles (if any) will cause a ball to travel equal distances, and evidence to support it.
- ▶ Angle(s) that will cause a ball to travel the farthest distance possible.



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## Index

2. A raft is being moved by a wind blowing east at 20 m/s and a current flowing south at 30 m/s.

- What is the raft's position after 8 s relative to its starting position?
- What equations simulate this motion?

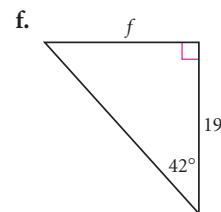
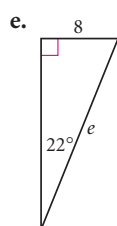
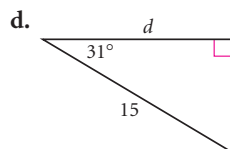
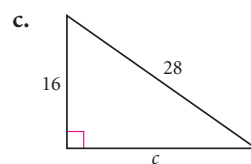
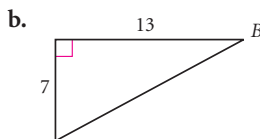
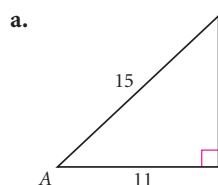
3. Graph each pair of parametric equations in a friendly window with factor 2, then eliminate the parameter to get a single equation using only  $x$  and  $y$ . Graph the resulting equation and describe how it compares with the original graph.

- $x = 2t - 5$  and  $y = t + 1$
- $x = t^2 + 1$  and  $y = t - 2$ ,  $-2 \leq t \leq 6$
- $x = \frac{t+1}{2}$  and  $y = t^2$ ,  $-4 \leq t \leq 3$
- $x = \sqrt{t+2}$  and  $y = t - 3$

4. Write parametric equations that will result in each transformation below for the equations  $x = 2t - 5$  and  $y = t + 1$ .

- Reflect the curve across the  $y$ -axis.
- Reflect the curve across the  $x$ -axis.
- Translate the curve up 3 units.
- Translate the curve left 4 units and down 2 units.

5. For each triangle, find the measure of the labeled angle or the length of the labeled side.



6. Sketch a graph of  $x = t \cos 28^\circ$  and  $y = t \sin 28^\circ$ . What is the angle between the graph and the  $x$ -axis?



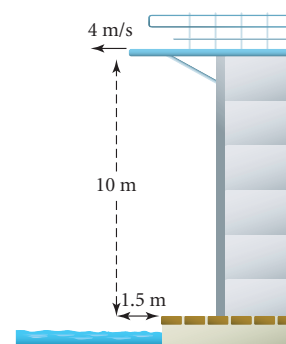
*The Family* (1962) by Venezuelan sculptor and painter Marisol Escobar (b 1930) has individual panels that collectively contribute to the whole piece, just as parametric equations combine individual equations to describe one mathematical piece.

7. A diver runs off a 10 m platform with an initial horizontal velocity of 4 m/s. The edge of the platform is directly above a point 1.5 m from the pool's edge. Simulate her motion on your graphing calculator. How far from the edge of the pool will she hit the water?

### Sports

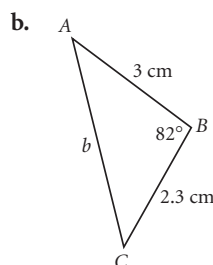
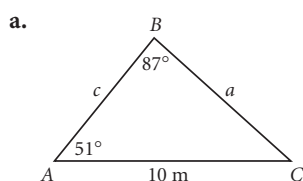
#### CONNECTION

In competitive diving, a running dive must be at least four steps long. The takeoff phase determines the diver's path through the air. Once in the air, the diver has less than 2 seconds to finish the dive, which should end with the diver's body almost perpendicular to the water's surface.





A diver competes in the women's 10 m platform competition in the 2000 Olympic Games in Sydney, Australia.

8. A duck paddles at a rate of 2.4 ft/s, aiming directly for the opposite bank of a 47 ft wide river. When he lands, he finds himself 28 ft downstream from the point where he started. What is the speed of the current?
9. Aliya sees a coconut 66 ft up in a coconut palm that is 20 ft away from her. She has a slingshot capable of launching a rock at 100 ft/s. If she launches a rock at an angle of  $72^\circ$ , will she hit the coconut? If not, by how much will she miss it?
10. **APPLICATION** A pilot is flying to a destination 700 mi away at a bearing of  $105^\circ$ . The cruising speed of the plane is 500 mi/h, and the wind is blowing between 20 mi/h and 30 mi/h at a bearing of  $30^\circ$ . At what bearing should she aim the plane to compensate for the wind?
11. Use the Law of Sines and the Law of Cosines to find all of the missing side lengths and angle measures.



## TAKE ANOTHER LOOK

1. In previous chapters you made connections between equations and their graphs. For example, the graph of an equation in the form  $y = a + bx$  is a line, and the graph of an equation in the form  $y = ax^2 + bx + c$  is a parabola. Can you make similar connections for parametric equations? What type of graph results if the parametric equation for  $x$  is quadratic and the parametric equation for  $y$  is linear? If both parametric equations are quadratic? Experiment with different combinations of parametric equations for  $x$  and  $y$ , and make conjectures about the resulting graphs.
2. You can use Boolean expressions to simulate motion that changes at a particular time. (See the project on page 232 for a description of Boolean expressions.) For example, Todd walks 3 mi/h on a bearing of  $210^\circ$  for 2.5 h, then turns and walks due east for 1.75 h. He then returns directly to his starting point. What parametric Boolean equations model this scenario? Include a sketch of your graph, including the window used.  See **Calculator Note 8H** to see how to graph Boolean expressions. 
3. Select a pair of noncongruent angles that are supplementary (whose measures sum to  $180^\circ$ ). Use your calculator to find the sine, cosine, and tangent of each angle measure. What relationships do you notice? Try other supplementary pairs to verify your relationships. Then select a pair of complementary angles (whose measures sum to  $90^\circ$ ), and find the sine, cosine, and tangent of each angle measure. What relationships do you notice? Verify these relationships with other complementary pairs. Draw geometric diagrams that prove the relationships you find.

## Assessing What You've Learned



**WRITE IN YOUR JOURNAL** In this chapter you used parametric equations to model several different real-world situations, such as the motion of an airplane traveling on a bearing and the motion of a projectile under the influence of gravity. These two situations are physically different, but you can use parametric equations to model both types of motion. In what ways are the two situations different? Are there any physical similarities between the two situations? How do the resulting parametric equations represent the similarities and differences between the two types of motion?



**ORGANIZE YOUR NOTEBOOK** Make sure your notebook is complete and well organized. Be sure to include all of the definitions and formulas that you have learned in this chapter. You may want to include at least one example of each of the different applications of parametric equations and trigonometry.



**WRITE TEST ITEMS** Write at least four test items for this chapter. Include items that cover different applications of both parametric equations and trigonometry. You may want to include the use of parametric equations for motion on a bearing and projectile motion, and the use of trigonometry for right triangles and oblique triangles.