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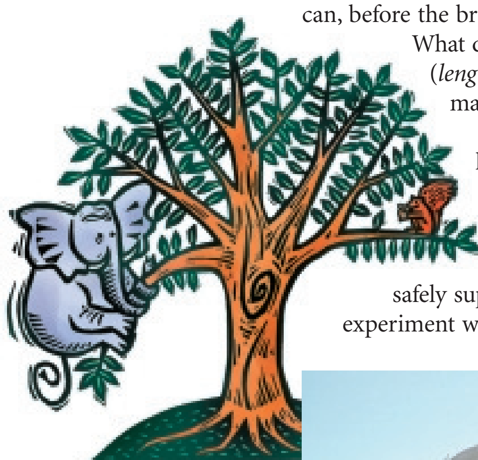
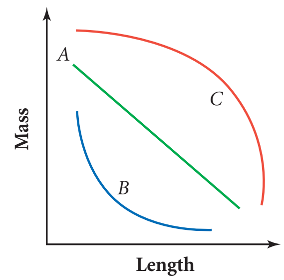
LESSON 9.6

Introduction to Rational Functions

You probably know that a lighter tree climber can crawl farther out on a branch than a heavier climber can, before the branch is in danger of breaking. What do you think the graph of (*length, mass*) data will look like when mass is added to a length of pole until it breaks? Is the relationship linear, like line *A*, or does it resemble one of the curves, *B* or *C*?





Engineers study problems like this because they need to know the weight that a beam can safely support. In the next investigation you will collect data and experiment with this relationship.

The Louise M. Davies Symphony Hall, built in 1980, is part of the Civic Center in San Francisco, California. The design of both the balcony and the covered entrance rely upon cantilevers—projecting beams that are supported at only one end.

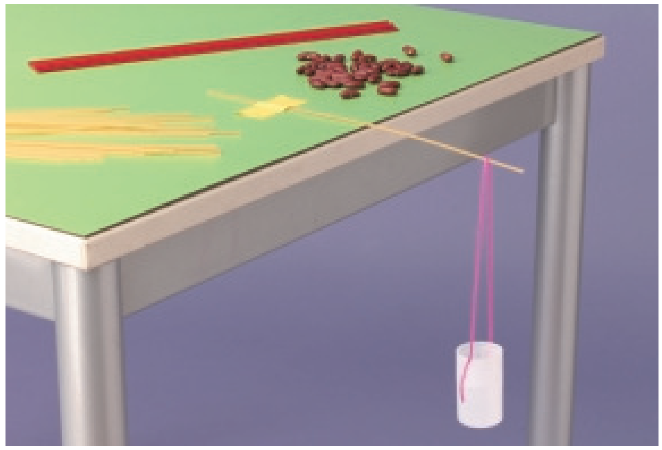
Investigation
The Breaking Point

You will need

- several pieces of dry spaghetti
- a small film canister
- string
- some weights (pennies, beans, or other small units of mass)
- a ruler
- tape

Procedure Note

1. Lay a piece of spaghetti on a table so that its length is perpendicular to one side of the table and the end extends over the edge of the table.
2. Measure the length of the spaghetti that extends beyond the edge of the table. (See the photo on the next page.) Record this information in a table of (*length, mass*) data.
3. Tie the string to the film canister so that you can hang it from the end of the spaghetti. (You may need to use tape to hold the string in place.)
4. Place mass units into the container one at a time until the spaghetti breaks. Record the maximum number of weights that the length of spaghetti was able to support.



Step 1 Work with a partner. Follow the procedure note to record at least five data points and then compile your results with those of other group members.

Step 2 Make a graph of your data with length as the independent variable, x, and mass as the dependent variable, y. Does the relationship appear to be linear? If not, describe the appearance of the graph.

Step 3 Write an equation that is a good fit with the plotted data.

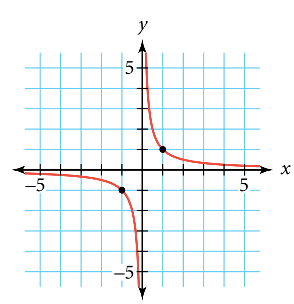
The relationship between length and mass in the investigation is an **inverse variation.** The parent function for an inverse variation curve, $f(x) = \frac{1}{x}$, is the simplest **rational function.**

Rational Function

A **rational function** is one that can be written as a quotient, $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomial expressions. The denominator polynomial must be of degree 1 or higher.

This type of function can be transformed just like all the other functions you have previously studied. The function you found in the investigation was probably a transformation of $f(x) = \frac{1}{x}$.

Graph the function $f(x) = \frac{1}{x}$ on your calculator and observe some of its special characteristics. The graph is made up of two branches. One part occurs where x is negative and the other where x is positive. There is no value for this function when $x = 0$. What happens when you try to evaluate $f(0)$? Notice that the graph is a hyperbola that has been rotated 45°, and has vertices (1, 1) and $(-1, -1)$. The x- and y-axes are the asymptotes. As x gets closer to zero, the y-values become increasingly large in absolute value.



Consider these values of the function $f(x) = \frac{1}{x}$.

x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
y	-1	-10	-100	-1000	undefined	1000	100	10	1

As x approaches zero from the negative side, the y-values have an increasingly large absolute value.

So $x = 0$ is a vertical asymptote.

As x approaches zero from the positive side, the y-values have an increasingly large absolute value.

The behavior of the y-values as x gets closer to zero shows that the y-axis is a vertical asymptote for this function.

x	-10000	-1000	-100	-10	0	10	100	1000	10000
y	-0.0001	-0.001	-0.01	-0.1	undefined	0.1	0.01	0.001	0.0001

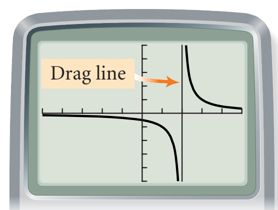
As x takes on larger negative values, the y-values approach zero.

So $y = 0$ is a horizontal asymptote.

As x takes on larger positive values, the y-values approach zero.

As x approaches the extreme values at the left and right ends of the x-axis, the curve approaches the y-axis. The horizontal line $y = 0$, then, is a horizontal asymptote. This asymptote is called an end behavior model of the function. In general, the end behavior of a function is its behavior for x-values that are large in absolute value.

If you think of $y = \frac{1}{x}$ as a parent function, then $y = \frac{1}{x} + 1$, $y = \frac{1}{x-2}$, and $y = 3\left(\frac{1}{x}\right)$ are examples of transformed rational functions. What happens to a function when x is replaced with $(x - 2)$? The function $y = \frac{1}{x-2}$ is shown at right.



$[-5, 5, 1, -5, 5, 1]$

Frequently, rational function graphs on the calculator include a nearly vertical drag line. The drag line is not part of the graph! However, it will look much like the graph of the vertical asymptote. [▶ See **Calculator Note 9C** to learn how to eliminate this line from your graph. ◀]

EXAMPLE A

Describe the function $f(x) = \frac{2x-5}{x-1}$ as a transformation of the parent function, $f(x) = \frac{1}{x}$. Then sketch a graph.

▶ Solution

You can change the form of the equation so that the transformations are more obvious. Because the denominator is $(x - 1)$, rather than x, try to get the expression $(x - 1)$ in the numerator as well.

$$f(x) = \frac{2x-5}{x-1}$$

Original equation.

$$f(x) = \frac{2(x-1)-3}{x-1}$$

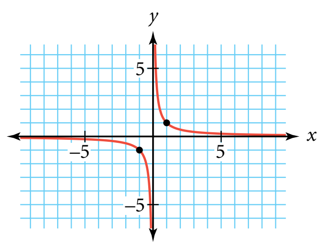
Consider the numerator to be $2x - 2 - 3$, and then factor to get $2(x - 1) - 3$.

Now you want to look for a scale factor and an added term. Separate the rational expression into two fractions:

$$f(x) = \frac{2(x-1)}{x-1} - \frac{3}{x-1}$$ Separate the numerator into two numerators over the same denominator.

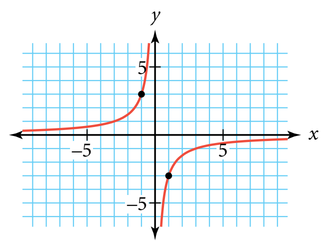
$$f(x) = 2 - \frac{3}{x-1}$$ Reduce.

Now you can see that the parent function has been vertically stretched by a factor of -3, then translated right 1 unit and up 2 units.



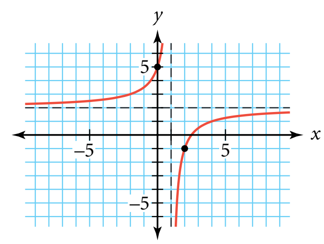
$y = \frac{1}{x}$

The parent rational function, $y = \frac{1}{x}$, has vertices $(1, 1)$ and $(-1, -1)$.



$y = -\frac{3}{x}$

A vertical stretch of -3 moves the vertices to $(1, -3)$ and $(-1, 3)$. The points $(3, 1)$ and $(-3, -1)$ are also on the curve. Notice that this graph looks more "spread out" than the graph of $y = \frac{1}{x}$.

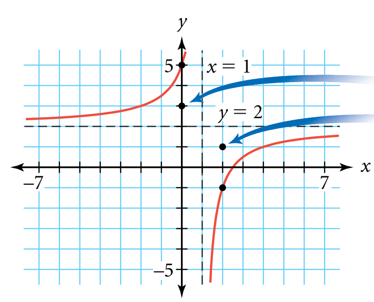


$y = 2 - \frac{3}{x-1}$

A translation right 1 unit and up 2 units moves the vertices to $(2, -1)$ and $(0, 5)$. The asymptotes are also translated to $x = 1$ and $y = 2$.

Notice that the asymptotes have been translated. How are the equations of the asymptotes related to your final equation above?

To identify an equation that will produce a given graph, do the procedure you used to graph Example A in reverse. You can identify translations by simply looking at the translations of the asymptotes. To identify stretch factors, pick a point, such as a vertex, whose coordinates you would know after the translation of $f(x) = \frac{1}{x}$. Then find a point on the stretched graph that has the same x-coordinate. The ratio of the vertical distances from the horizontal asymptote to those two points is the vertical scale factor.



An unstretched (but reflected) inverse variation function with these translations would have vertices $(2, 1)$ and $(0, 3)$, 1 horizontal unit and 1 vertical unit away from the center. Because the distance is now 3 vertical units from the center, include a vertical stretch of 3 to get the equation

$$y = 2 - \frac{3}{x-1}$$

Rational expressions are very useful in chemistry. Scientists use them to model many situations, including the concentration of a solution or mixture as it is diluted.

EXAMPLE B Suppose you have 100 mL of a solution that is 30% acid and 70% water. How many mL of acid do you need to add to make a solution that is 60% acid? To make it a 90% acid solution? Can it ever be 100% acid?

▶ **Solution** Of the 100 mL of solution, 30%, or 30 mL, is acid. The percentage, P, can be written as $P = \frac{30}{100}$. If x milliliters of acid are added, there will be more acid, but also more solution. The concentration of acid will be

$$P = \frac{30 + x}{100 + x}$$

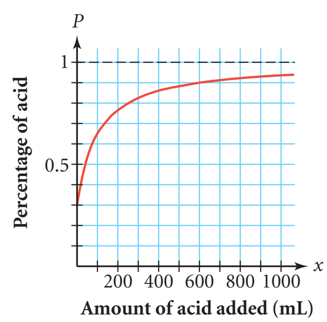
To find when the solution is 60% acid, substitute 0.6 for P and solve the equation.

$0.6 = \dfrac{30 + x}{100 + x}$	Substitute 0.6 for P.
$0.6(100 + x) = 30 + x$	Multiply both sides by $(100 + x)$.
$60 + 0.6x = 30 + x$	Distribute.
$30 = 0.4x$	Collect like terms.
$75 = x$	Divide by 0.4.

Adding 75 mL of acid will make a 60% acid solution.

To find when the solution is 90% acid, solve the equation $0.9 = \frac{30 + x}{100 + x}$. You will find that 600 mL of acid must be added.

The graph of $P = \frac{30 + x}{100 + x}$ shows horizontal asymptote $y = 1$. No matter now many milliliters of acid you add, you will never have a mixture that is 100% acid. This is because the original 70 mL of water will remain, even though it is a smaller and smaller percentage of the entire solution as you continue to add acid.



EXERCISES

You will need

Geometry software for Exercise 13

Practice Your Skills

1. Write an equation and graph each transformation of the parent function $f(x) = \frac{1}{x}$.

a. Translate the graph up 2 units.

b. Translate the graph right 3 units.

c. Translate the graph down 1 unit and left 4 units.

d. Vertically stretch the graph by a scale factor of 2.

e. Horizontally stretch the graph by a factor of 3, and translate it up 1 unit.

2. What are the equations of the asymptotes for each hyperbola?

a. $y = \dfrac{2}{x} + 1$ **b.** $y = \dfrac{3}{x - 4}$ **c.** $y = \dfrac{4}{x + 2} - 1$ **d.** $y = \dfrac{-2}{x + 3} - 4$

3. Solve.

a. $12 = \dfrac{x-8}{x+3}$ **b.** $21 = \dfrac{3x+8}{x-5}$ **c.** $3 = \dfrac{2x+5}{4x-7}$ **d.** $-4 = \dfrac{-6x+5}{2x+3}$

4. As the rational function $y = \frac{1}{x}$ is translated, its asymptotes are translated also. Write an equation for the translation of $y = \frac{1}{x}$ that has the asymptotes described.

a. horizontal asymptote $y = 2$ and vertical asymptote $x = 0$

b. horizontal asymptote $y = -4$ and vertical asymptote $x = 2$

c. horizontal asymptote $y = 3$ and vertical asymptote $x = -4$

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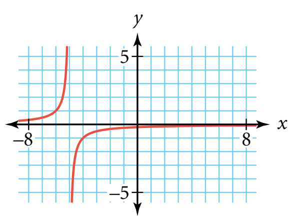
5. If a basketball team's present record is 42 wins and 36 losses, how many consecutive games must it win so that its winning record reaches 60%?

Reason and Apply

6. Write a rational equation to describe each graph. Some equations will need scale factors.

a.



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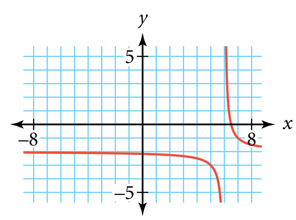

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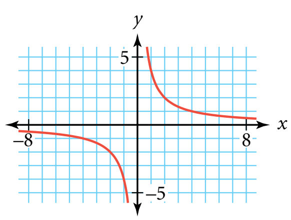

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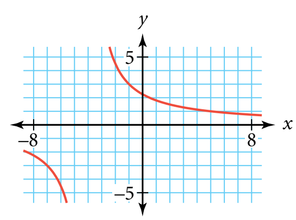
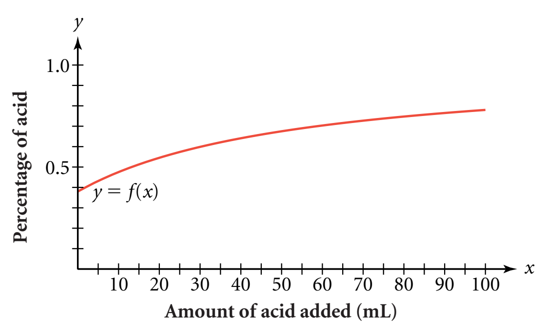


7. APPLICATION The graph at right shows the concentration of acid in a solution as pure acid is added. The solution began as 55 mL of a 38% acid solution.



a. How many milliliters of pure acid were in the original solution?

b. Write an equation for $f(x)$.

c. Find the amount of pure acid that must be added to create a solution that is 64% acid.

d. Describe the end behavior of $f(x)$.

8. APPLICATION In a container of 2% milk, 2% of the mixture is fat. How much of the liquid in a 1 gal container of 2% milk would need to be emptied and replaced with pure fat so that the container could be labeled as whole (3.25%) milk?

9. Consider these functions.

i. $y = \dfrac{2x - 13}{x - 5}$ **ii.** $y = \dfrac{3x + 11}{x + 3}$

a. Rewrite each rational function to show how it is a transformation of $y = \frac{1}{x}$.

b. Describe the transformations of the graph of $y = \frac{1}{x}$ that will produce graphs of the equations in 9a.

c. Graph each equation on your calculator to confirm your answers to 9b.

10. Draw the graph of $y = \frac{1}{x}$.

a. Label the vertices of the hyperbola.

b. The x- and y-axes are the asymptotes for this hyperbola. Draw the box between the two branches of the hyperbola that has the asymptotes as its diagonals. The vertices should lie on the box.

c. The dimensions of the box are $2a$ and $2b$. Find the values of a and b.

d. The foci of the hyperbola lie on the line passing through the two vertices. In this case, that line is $y = x$. The foci are c units from the center of the hyperbola where $a^2 + b^2 = c^2$. Find the value of c and the coordinates of the two foci.

11. Recall that the general quadratic equation is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Let $A = 0$, $B = 4$, $C = 0$, $D = 0$, $E = 0$, and $F = -1$.

a. Graph this equation. What type of conic section is formed?

b. What is the relationship between the inverse variation function, $y = \frac{1}{x}$, and the conic sections?

c. Convert the rational function $y = \frac{1}{x-2} + 3$ to general quadratic form. What are the values of A, B, C, D, E, and F in the general quadratic equation?

12. **APPLICATION** Ohm's law states that $I = \frac{V}{R}$. This law can be used to determine the amount of current I, in amps, flowing in the circuit when a voltage V, in volts, is applied to a resistance R, in ohms.

a. If a hairdryer set on high is using a maximum of 8.33 amps on a 120-volt line, what is the resistance in the heating coils?

b. In the United Kingdom, power lines use 240 volts. If a traveler were to plug in a hairdryer, and the resistance in the hairdryer was the same as in 12a, what would be the flow of current?

c. The additional current flowing through the hairdryer would cause a meltdown of the coils and the motor wires. In order to reduce the current flow in 12b back to the value in 12a, how much resistance would be needed?

Consumer CONNECTION

Many travel appliances, such as hairdryers and shavers, are made with a voltage switch that provides the resistance necessary for the appliance to work properly in different countries. In the United States, a voltage of about 120V is standard, whereas in Europe about 240V is typical. A dimmer switch on a light fixture works in a similar way. When the dimmer switch is set low, there is higher resistance, causing less current to flow, and less illumination is produced. As the dimmer switch is turned, there is less resistance, allowing more current to flow, and more illumination is produced. The volume control on a stereo system works the same way.

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13. *Technology* Using geometry software, draw the curve $y = \frac{1}{x}$ and plot the foci you found in Exercise 10d. Use measurement tools to verify that $y = \frac{1}{x}$ satisfies the locus definition of a hyperbola.

Review

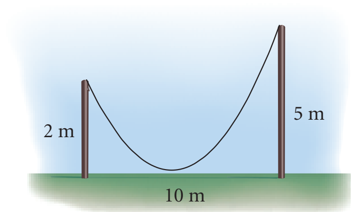
14. Factor each expression completely.

a. $x^2 - 7x + 10$

b. $x^3 - 9x$

15. Write the equation of the circle with center $(2, -3)$ and radius 4.

16. A 2 m rod and a 5 m rod are mounted vertically 10 m apart. One end of a 15 m wire is attached to the top of each rod. Suppose the wire is stretched taut and fastened to the ground between the two rods. How far from the base of the 2 m rod is the wire fastened?

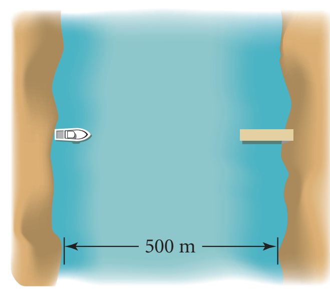


17. Sarah would like to row her boat directly across a river 500 m wide. The current flows 3 km/h and she is able to row 5 km/h.

a. At what angle to the riverbank should she point her boat?

b. As she starts, how far upstream on the opposite bank should she head?

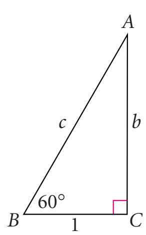
c. Write parametric equations to simulate Sarah's crossing.



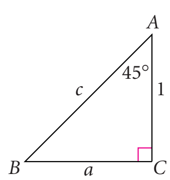
18. Write the general quadratic equations of two concentric circles with center $(6, -4)$ and radii 5 and 8.

19. Find exact values of missing side lengths for 19a–d.

a.



b.



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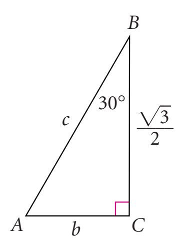
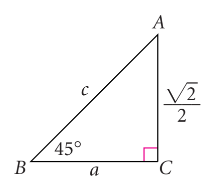


d.



e. What is the ratio of side lengths in a 45°-45°-90° triangle? A 30°-60°-90° triangle?